Notes of the Course Graphics

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December 14, 2018

*These notes are incomplete and not reviewed. If someone wants to help, mail me at giovanni.barbarino@gmail.com, and you'll be included in the list of authors.

Contents

1 Totally Positive Matrices and Sets

Definition 1 (TP)**.** A matrix $A \in \mathbb{R}^{n\times m}$ is called **Totally Positive** if the determinant of every square submatrix is positive.

Definition 2 (TN)**.** A matrix $A \in \mathbb{R}^{n\times m}$ is called **Totally Nonnegative** if the determinant of every square submatrix is nonnegative.

A property of both the classes is that they're closed under multiplication. In fact if A, B are TP, then AB is TP and the same holds for TN matrices.

An example of these kind of matrices is the Vandermonde Matrix of $x_1, \dots, x_n \in \mathbb{R}$.

$$V(x_1,\dots,x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} \qquad V(x_1,\dots,x_n)\begin{pmatrix} c_0 \\ \vdots \\ c_{n-1}\end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n-1} c_i x_1^i \\ \vdots \\ \sum_{i=0}^{n-1} c_i x_n^i \end{pmatrix}$$

$$\det V(x_1,\dots,x_n) = \prod_{1\le i<j\le n} x_j - x_i$$

In fact, any Vandermonde matrix referred to positive and strictly increasing x_i is totally positive.

Lemma 1. *Given $0 < x_1 < x_2 < \dots < x_n$ real numbers, the matrix $V(x_1,\dots,x_n)$ is totally positive*

Proof. Let A be a square submatrix

$$A = \begin{pmatrix} x_{l_1}^{\alpha_1} & \dots & x_{l_1}^{\alpha_k} \\ \vdots & & \vdots \\ x_{l_k}^{\alpha_1} & \dots & x_{l_k}^{\alpha_k} \end{pmatrix}$$

with $l_1 < l_2 < \dots < l_k$ and $\alpha_1 < \alpha_2 < \dots < \alpha_k$. Given the polynomial

$$p(x) = c_1 x^{\alpha_1} + \dots + c_k x^{\alpha_k}$$

with $c_i \in \mathbb{R}$, then we know thanks to Cartesio that the number of real positive roots of $p(x)$ is less then the number of sign changes in the coefficients, so it is at most $k-1$. If we want to solve $p(x_{l_i}) = 0$ for every i, then it is equivalent to

$$A\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = 0.$$

If there are solutions other than $c_1 = \dots = c_k = 0$, then $p(x)$ has at least k positive solutions $x_{l_1}, \dots, x_{l_k}$ that is an absurd. So A is invertible and its determinant is not zero. Let's compute the determinant of A starting from the last row.

$$\det A = x_{l_k}^{\alpha_k} a_k - x_{l_k}^{\alpha_{k-1}} a_{k-1} + \dots + (-1)^{k-1} x_{l_k}^{\alpha_1} a_1 = g(x_{l_k}) \qquad a_k = \begin{pmatrix} x_{l_1}^{\alpha_1} & \dots & x_{l_1}^{\alpha_{k-1}} \\ \vdots & & \vdots \\ x_{l_{k-1}}^{\alpha_1} & \dots & x_{l_{k-1}}^{\alpha_{k-1}} \end{pmatrix}, \; a_{k-1} = \dots$$

Let's proceed by induction on k, proving that the submatrices have strictly positive determinant. If $k = 1$ it is easy. Otherwise, in the last relation, we know that $a_i > 0$ for every i. It means that $g(x) \to +\infty$ if $x \to +\infty$, and $g(x) \neq 0$ for every $x > x_{l_{k-1}}$ since $\det(A) \neq 0$. So $\det A = g(x_{l_k}) > 0$.

□

The same proof could be a lot simpler using a result saying that A is TP if and only if the determinant of square submatrices composed only by contiguous rows and columns are positive.

Definition 3 (Bidiagonal)**.** A matrix $A \in \mathbb{R}^{n\times n}$ is called **Bidiagonal** if there exists an $s \in \mathbb{Z}$ such that $A_{i,j} = 0$ whenever $i - j < s$ or $i - j > s + 1$.
An **Upper Bidiagonal** has $s = -1$, and a **Lower Bidiagonal** has $s = 0$.

Theorem 1 (Factorization)**.** *A matrix A is TN if and only if it can be factorized into a product of upper and lower bidiagonal TN matrices with at most one element outside the main diagonal.*

We won't report the proof. A reference for further infos is [1].
We report as an example

$$P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} \qquad (P_n)_{i,j} = \binom{i+j-2}{i-1}$$

Let us reduce the matrix through bidiagonal matrices operating on rows

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operating on the columns we can continue reducing (Neuvill Reduction) and if P_4 is TP we won't encounter zeros on the pivots, so the factorization can be carried out until the end. In this case, we are using bidiagonal matrices $E_i(\alpha)$ that is an identity matrix plus an α element in position $(i, i-1)$. Notice that $E_i(\alpha)^{-1} = E_i(-\alpha)$.

In general any TP matrix can be factorized into the product

$$\prod_i (E_{k_i}(\alpha_i)) D \prod_j (E^T_{s_j}(\alpha_j))$$

where D is a nonnegative diagonal matrix and $\alpha_i, \alpha_j > 0$. In the case TN, we have to worry about null pivots, but the final factorization is similar.

Lemma 2. *If A is a TN matrix, and $x \in \mathbb{R}^n$, then the number of sign changes in x is not less than the number of sign changes in Ax.*

Proof. If B is a bidiagonal TN matrix of the form $E_i(\alpha)$, then $V(Bx) \leq V(x)$, where $V(\cdot)$ counts the number of sign changes. In fact we have to check only what happens within the element x_i because the other elements don't change. In fact, $(Bx)_i = \alpha x_{i-1} + x_i$, so if x_i and x_{i-1} have the same sign, then $(Bx)_i$ has still the same sign of x_i. In any other case, changing the sign of x_i diminish the number of sign changes. The same holds with $B = E_i^T(\alpha)$ and if B is diagonal and nonnegative. We can thus factorize A and conclude that $V(Ax) \leq V(x)$. □

Definition 4 (TN/TP Set)**.** A set $\{\varphi_1, \dots, \varphi_p\}$ of functions on an interval $I \subseteq \mathbb{R}$ is said **Totally Nonnegative** or **Totally Positive** if every choice of points $t_1 < \dots < t_r$ inside I leads, respectively, to a TN or TP matrix

$$\begin{pmatrix} \varphi_1(t_1) & \varphi_2(t_1) & \dots & \varphi_p(t_1) \\ \vdots & \vdots & & \vdots \\ \varphi_1(t_r) & \varphi_2(t_r) & \dots & \varphi_p(t_r) \end{pmatrix}.$$

The set is **Normalized** if

$$\sum_{i=1}^{p} \varphi_i(x) = 1 \qquad \forall x \in I$$

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For example, Bernstein Polynomials are a normalized basis of the polynomial space.
The matrix in the definition is call **Collocation Matrix**, and if we multiply it by a vector, we obtain

$$\begin{pmatrix} \varphi_1(t_1) & \varphi_2(t_1) & \dots & \varphi_p(t_1) \\ \vdots & \vdots & & \vdots \\ \varphi_1(t_r) & \varphi_2(t_r) & \dots & \varphi_p(t_r) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^p \varphi_i(t_1)c_i \\ \vdots \\ \sum_{i=1}^p \varphi_i(t_r)c_i \end{pmatrix}$$

that are the evaluations of $\sum_{i=1}^p \varphi_i(x)c_i$ on $t_1, t_2, \dots, t_r$.

Lemma 3. *If $\{\varphi_0, \dots, \varphi_p\}$ is a TN set on I, then*

1. *given $f : J \to I$ strictly increasing function (where J is an interval), the composition $\{\varphi_0 \circ f, \dots, \varphi_p \circ f\}$ is a TN set for J,*
2. *given $g : I \to \mathbb{R}$ a nonnegative function, the set $\{g \cdot \varphi_0, \dots, g \cdot \varphi_p\}$ is a TN set for I,*
3. *if $A \in \mathbb{R}^{(p+1)\times(p+1)}$ is a TN nonsingular matrix, then*

$$\left\{ \sum_{j=1}^{p+1} A_{1,j}\varphi_{j-1}, \dots, \sum_{j=1}^{p+1} A_{p+1,j}\varphi_{j-1} \right\}$$

is a TN set on I.

Proof.
1) The collocation matrices of the new set are

$$\begin{pmatrix} \varphi_0(f(u_1)) & \varphi_1(f(u_1)) & \dots & \varphi_p(f(u_1)) \\ \vdots & \vdots & & \vdots \\ \varphi_0(f(u_r)) & \varphi_1(f(u_r)) & \dots & \varphi_p(f(u_r)) \end{pmatrix}$$

that is a collocation marix for the old set on the points $t_i = f(u_i)$, but f is strictly increasing, so $t_1 < \dots < t_r$ and the matrix is thus TN.
2) The collocation matrices of the new set are

$$\begin{pmatrix} g\varphi_0(t_1) & g\varphi_1(t_1) & \dots & g\varphi_p(t_1) \\ \vdots & \vdots & & \vdots \\ g\varphi_0(t_r) & g\varphi_1(t_r) & \dots & g\varphi_p(t_r) \end{pmatrix} = \begin{pmatrix} g(t_1) & & & \\ & g(t_2) & & \\ & & \ddots & \\ & & & g(t_r) \end{pmatrix} \begin{pmatrix} \varphi_0(t_1) & \varphi_1(t_1) & \dots & \varphi_p(t_1) \\ \vdots & \vdots & & \vdots \\ \varphi_0(t_r) & \varphi_1(t_r) & \dots & \varphi_p(t_r) \end{pmatrix}$$

that is a product of TN matrices.
3) The collocation matrices for the new set on the points $t_1, \dots, t_r$ is the product the collocation matrices for the old set and A^T, so it is a TN matrix.

$$\begin{pmatrix} \sum_{j=1}^{p+1} A_{1,j}\varphi_{j-1}(t_1) & \sum_{j=1}^{p+1} A_{2,j}\varphi_{j-1}(t_1) & \dots & \sum_{j=1}^{p+1} A_{p+1,j}\varphi_{j-1}(t_1) \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{p+1} A_{1,j}\varphi_{j-1}(t_r) & \sum_{j=1}^{p+1} A_{2,j}\varphi_{j-1}(t_r) & \dots & \sum_{j=1}^{p+1} A_{p+1,j}\varphi_{j-1}(t_r) \end{pmatrix} = \begin{pmatrix} \varphi_0(t_1) & \varphi_1(t_1) & \dots & \varphi_p(t_1) \\ \vdots & \vdots & & \vdots \\ \varphi_0(t_r) & \varphi_1(t_r) & \dots & \varphi_p(t_r) \end{pmatrix} A^T$$

□

Corollary 1. *Let $\{\varphi_0, \dots, \varphi_p\}$ be a TN basis for $\mathbb{P}_p$ (space of polynomials of degree $\leq p$) on I. If $c_0, \dots, c_p \in \mathbb{R}$, then*

$$V\left(\sum c_i\varphi_i\right) \leq V(c_0, \dots, c_p)$$

where V applied on a function counts the number of sign changes.

Proof. If the polynomial $f = \sum c_i\varphi_i$ changes its sign more than s times, where $s = V(c_0, \dots, c_p)$, then we can find $t_0, \dots, t_{s+1}$ points such that $V(f(t_0), \dots, f(t_{s+1})) = s + 1$, but

$$\begin{pmatrix} f(t_0) \\ \vdots \\ f(t_{s+1}) \end{pmatrix} = \begin{pmatrix} \varphi_0(t_0) & \varphi_1(t_0) & \dots & \varphi_p(t_0) \\ \vdots & \vdots & & \vdots \\ \varphi_0(t_{s+1}) & \varphi_1(t_{s+1}) & \dots & \varphi_p(t_{s+1}) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_p \end{pmatrix}$$

and this is an absurd thanks to Lemma 2. □

2 Shape Optimality

Definition 5 (Generalized Bezier Curves)**.** Given $\{\varphi_0, \dots, \varphi_p\}$ a basis of $\mathbb{P}_p$ and $c_0, \dots, c_p \in \mathbb{R}^d$, then

$$\mathscr{C}(t) = \sum_{j=1}^{p} c_i\varphi_i(t)$$

are called **Generalized Bexier Curves**, where c_i are the **Control Points**, whose envelop is called **Control Polygon**. In Numerical Graphics jargon, $\varphi_i(t)$ are called **Blending Functions**.

An example is Timmer's parametric cubic, with functions

$$f_0(t) = (1 - 2t)(1 - t)^2 \quad f_1(t) = 4t(1 - t)^2 \quad f_2(t) = 4t^2(1 - t) \quad f_3(t) = (2t - 1)t^2$$

that has the property of going through the middle point of P_1P_2 where P_0, P_1, P_2, P_3 are the control points. It loses the property of being contained in the convex envelop of the control points, but it gains a symmetry:

$$f_i(1 - t) = f_{3-i}(t) \qquad \forall i.$$

An other example is Ball's parametric cubic used in CAD

$$f_0(t) = (1 - t)^2 \quad f_1(t) = 2t(1 - t)^2 \quad f_2(t) = 2t^2(1 - t) \quad f_3(t) = t^2$$

In this case, if $P_1 = P_2$ then we have a quadratic curve, since $f_1 + f_2 = 2t(1 - t)$.

A last example is Overhauser curve, or Catmull-Rom splines, used at Ford. In this case, the resulting curve interpolates the control points.

Theorem 2 (Variation Diminishing Property)**.** *Let $\{\varphi_0, \dots, \varphi_p\}$ be a normalized TN basis of $\mathbb{P}_p$ on I, and let $c_0, \dots, c_p \in \mathbb{R}^2$ be the control points of the GBC $\mathscr{C}(t)$. If l is a line in $\mathbb{R}^2$, then the number of intersections between l and $\mathscr{C}(t)$ are less or equal than the number of intersections between l and the control polygon P.*

Proof. Let $c_j = (c_{jx}, c_{jy})$ and $\mathscr{C}(t) = (\mathscr{C}_x(t), \mathscr{C}_y(t))$ and let $ax+by+c=0$ be the equation of l. The intersections between l and $\mathscr{C}(t)$ are

$$\begin{aligned} 0 &= a\mathscr{C}_x(t) + b\mathscr{C}_y(t) + c & (1)\\ &= a\sum_{i=0}^{p} c_{ix}\varphi_i(t) + b\sum_{i=0}^{p} c_{iy}\varphi_i(t) + c\sum_{i=0}^{p}\varphi_i(t) & (2)\\ &= \sum_{i=0}^{p}\varphi_i(t)[ac_{ix} + bc_{iy} + c] & (3) \end{aligned}$$

By Corollary 1, we know that

$$V\left(\sum_{i=0}^{p}\varphi_i(t)[ac_{ix} + bc_{iy} + c]\right) \le V(ac_{0x} + bc_{0y} + c, \ldots, ac_{px} + bc_{py} + c)$$

that represent, respectively, the number of intersection of l with the Bezier Curve and the control polygon. □

The last theorem indicates that TN normalized bases are a "shape-preserving" representation of the curves. (Actually it holds if the line l is not tangent to the polygon or the curve).

An example of TN basis is the monomial basis $\{1, x, \ldots, x^p\}$ that gives us Vandermonde collocation matrices (see Lemma 1) on $[0, +\infty)$.

An other example is Bernstein basis. In fact, starting from the monomial basis, one can apply the strictly increasing function

$$f(t) = \frac{t}{1-t}$$

and multiply by the nonnegative function $g(t) = (1-t)^p$ obtaining

$$(1-t)^p\frac{t^j}{(1-t)^j} = (1-t)^{p-j}t^j.$$

Eventually, we multiply by the diagonal matrix

$$\operatorname{diag}\left(\binom{p}{0}, \binom{p}{1}, \ldots, \binom{p}{p}\right)$$

and Lemma 3 assures us that the resulting Bernstein basis

$$B_j(t) = \binom{p}{j}(1-t)^p\frac{t^j}{(1-t)^j} = \binom{p}{j}(1-t)^{p-j}t^j.$$

is a TN set on $[0, +\infty)$.

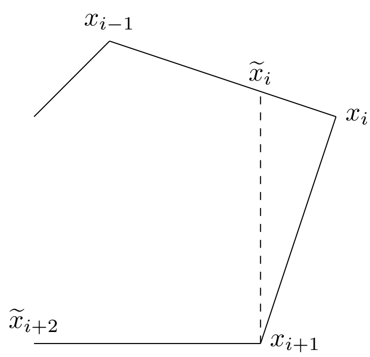
The Bernstein basis is "geometrically optimal", as we'll explain in a moment.
Given two normalized TN basis $\{\varphi_0, \ldots, \varphi_p\}$ and $\{\theta_0, \ldots, \theta_p\}$ of $\mathbb{P}_p$ on I, suppose there exists a TN matrix K such that

$$(\varphi_0, \ldots, \varphi_p) = (\theta_0, \ldots, \theta_p) \cdot K.$$

K must be invertible and row-stochastic, since they are both normalized basis and K is nonnegative. In this case, K can be factorized into bidiagonal row-stochastic matrices of the form

$$\begin{pmatrix} \ddots & & & \\ & 1 & & \\ & 1-\lambda_i & \lambda_i & \\ & & & \ddots \end{pmatrix}, \qquad \begin{pmatrix} \ddots & & & \\ & \mu_j & 1-\mu_j & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

If we apply these matrices to a vector, they only substitute one element with a convex combination of two consecutive entries.



So the new control polygon is smaller, and it is closer to the Bezier curve.

Bernstein polynomial basis has the property that any other TN normalized basis can be written as the Bernstein basis multiplied by a TN invertible matrix K, so Bernstein representation is geometrically optimal.

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3 Splines

Splines functions are useful for the problem of *Polynomial Interpolation*: given $f : [a, b] \to \mathbb{R}$ and interpolation nodes $a = u_0 < u_1 < \cdots < u_n = b$, we know that there exists exactly one polynomial of degree equal or less than n such that $p(u_i) = f(u_i)$ for every i.

In general, the resulting polynomial p does not approximate well f on the whole interval $[a, b]$, since it tends to oscillate too much (Runge phenomenon with $f(x) = 1/(1+25x^2)$ in $[-1, 1]$). A possible solution is to choose a non-regular grid with nodes that accumulate on the extrema of the interval (Chebychev). However, for every sequence of grids there always is a continuous function such that the polynomial interpolations on those grid do not converge to the original function (Faber Theorem).

Some positive notes: when dealing with Lipschitz functions, one can interpolate through the Chebychev nodes scheme and obtain a convergent approximation (Chebfun Project on Matlab).

High degree polynomials are also difficult to handle, so one can try with piecewise polynomial interpolation instead. In particular, one can look for low degree polynomial interpolations on each couple of nodes that behave well on the interfaces. In particular, we are looking for $s : [a, b] \to \mathbb{R}$ such that $s|_{[u_i, u_{i+1}]}$ is a polynomial of degree $< k$ with $s(u_i) = f(u_i)$ for every i. Since we have $k - 2$ free parameters for each polynomial, we can add conditions.

For example, *Hermite Condition* with $k = 4$ requires that s is C^1 and that s' interpolates f', meaning that $s'(u_i) = f'(u_i)$. When it is not easy to compute $f'(u_i)$ one can use *Bessel method* to approximate it with the derivative of the cubic function obtained interpolating u_i and its neighbors.

The **Cubic Splines** are C^2 functions, piecewise cubic, interpolating f. In particular, if $s_i = s|_{[u_i, u_{i+1}]}$, then s_i is a cubic (so 4 parameters for every i) and

$$\begin{aligned} s'_{i-1}(u_i) &= s'_i(u_i) && n-1 \text{ conditions} \\ s''_{i-1}(u_i) &= s''_i(u_i) && n-1 \text{ conditions} \\ s_i(u_i) &= f(u_i) && n \text{ conditions} \\ s_i(u_{i+1}) &= f(u_{i+1}) && n \text{ conditions} \end{aligned}$$

It leaves out 2 free parameters, so we can impose further conditions that distinguish different types of splines.

In general we can define them for every order k.

Definition 6. $f : [a, b] \to \mathbb{R}$ and interpolation nodes $a = u_0 < u_1 < \cdots < u_n = b$, the interpolating **k-th Order Spline** is a piecewise polynomial C^2 function $s : [a, b] \to \mathbb{R}$ such that $s_i = s|_{[u_i, u_{i+1}]}$ is a polynomial of degree $< k$ for every i.

Focusing on the cubic splines, that is $k = 4$, we can choose the last two free conditions to be

- $s'(a) = f'(a)$, $s'(b) = f'(b)$, called *Complete Splines.*
- $s''(a) = 0$, $s''(b) = 0$, called *Natural Splines*, but the convergence rate is slow (also called variational splines).
- $s_0'''(u_1) = s_1'''(u_1)$, $s_{n-1}'''(u_{n-1}) = s_n'''(u_{n-1})$, called *not-a-knot Splines.*
- If f is periodic, then $s'(a) = s'(b)$, $s''(a) = s''(b)$, called *Periodic Spline.*

The convergence, in the case of Complete Splines, is $O(h^4)$, where $h = \max_i\{|u_{i+1} - u_i|\}$.

Splines are interesting even because they respect the property of minimal curvature, meaning that the complete spline minimizes

$$\int_a^b |s''(x)|^2 dx$$

among all C^2 functions s interpolating f on the same nodes. This property assures us that the oscillating phenomenon observed with polynomial interpolation is minimized using splines interpolation.

The computational complexity for the solution is $O(n)$ since the resulting linear system is tridiagonal and diagonal dominant. Moreover, it is also well-conditioned.

We could also use Bezier curves to obtain a spline interpolation. Given the points P_0, P_1, P_2, P_3, where $P_0 = (u_0, f(u_0))$ and $P_3 = (u_1, f(u_1))$, we find a first curve. The choice of successive control points is determined by the regularity of the interpolation, since we want, for example, P_2P_3 parallel to P_3P_4 in order to have the same derivative at u_1.

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We'd like to have a basis $\{\varphi_i(t)\}_i$ such that the support of $\varphi_i(t)$ is small inside $I = [a, b]$, so that a 'small adjustment' of the curves is realized changing few elements (functions or control points). This is the main reason to develop the theory of **B-Splines**, that have been defined in several ways. We will use the difinition of De Boor, Cox and Mansfields by recurrent relations.

Choose $u_0, \dots, u_n$ a sequence of nondecreasing nodes (real numbers), where each node may be repeated. We refer to 'nodes' when we consider multiplicities, and 'break points' when we don't consider multiplicities.

Definition 7. The ***i*-th B-Spline of degree** p (order $p+1$) are denoted by $N_{i,p}(u)$ and are defined by recurrence

$$N_{i,0}(u) = \begin{cases} 1 & u_i \le u < u_{i+1} \\ 0 & otherwise \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

where a zero denominator is considered as zero.

In the definition, when computing $N_{i,p}$, we consider the nodes

$$u_0 = \dots = u_p \le u_{p+1} \le \dots \le u_{n+p-1} \le u_{n+p} = \dots = u_{n+2p}$$

to always obtain n functions $N_{0,p}, \dots, N(n-1, p)$.

The B-splines have useful properties:

- $N_{i,p}$ has support in $[u_i, u_{i+p+1}]$
- $N_{i,p}$ is in $C^\infty(u_j, u_{j+1})$ (since it is polynomial) for every j such that $u_j \neq u_{j+1}$ and C^{p-k} on knots of multiplicity k
- For every i there are at most $p+1$ functions $N_{i,p}$ that are not zero on $[u_i, u_{i+1}]$, namely $N_{i-p,p}, \dots, N_{i,p}$

- For every p, the set $\{ N_{i,p} \}_i$ is a partition of unity: they are all nonnegative, and

$$\sum_{j=i-p}^{i} N_{j,p}(u) = 1 \qquad \forall\, u \in [u_i, u_{i+1}]$$

(on the curves it corresponds to affine invariance)

- It is easy to compute derivatives of B-splines through recurrence relations
- They form a basis for piecewise polynomial functions $P\mathbb{P}^p$ on the breakpoints. They are defined as the functions s such that $s|_{[u_i,u_{i+1}]}$ are polynomials of degree p.

Exercise 1. *Consider the nodes* $0,0,0,1,1,1$ *and compute the B-splines for* $p = 0,1,2$.

$$N_{0,0} = N_{1,0} = N_{3,0} = N_{4,0} = 0 \qquad N_{2,0} = 1$$
$$N_{0,1} = N_{3,1} = 0 \qquad N_{1,1} = N_{2,0}\frac{1-u}{1} = 1-u \qquad N_{2,1} = N_{2,0}\frac{u}{1} = u$$
$$N_{0,2} = N_{1,1}\frac{1-u}{1} = (1-u)^2 \qquad N_{1,2} = N_{1,1}\frac{u}{1} + N_{2,1}\frac{1-u}{1} = 2u(1-u) \qquad N_{2,2} = N_{2,1}\frac{u}{1} = u^2$$

They remind us of Bernstein Polynomials. In general it is possible to compute Bernstein Polynomial by the B-splines recurrence relations using the knots $0,0,\dots,0,1,\dots,1,1$ where each knot has multiplicity $k+1$. The use of these Splines as blending functions generalizes the Bezier-Bernstein curves.

Let us prove some of the properties.

Nonnegativity. The proof is by induction. It is obvious for $N_{i,0}$ since they take values in $\{0,1\}$. Given a generic $p > 0$, we use the recurrence relation and the inductive hypothesis

$$N_{i,p}(u) = \frac{u-u_i}{u_{i+p}-u_i}N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}N_{i+1,p-1}(u)$$

Notice that $N_{i,p-1}$ has support in $[u_i, u_{i+p}]$, so $\frac{u-u_i}{u_{i+p}-u_i}N_{i,p-1}(u)$ is zero if $u < u_i$, and otherwise the whole term is nonnegative. A similar argument is used for the second term, so the whole function $N_{i,p}$ is sum of nonnegative functions.

Basis. Let us consider the set V of functions in $P\mathbb{P}^p$ on the knots $u_0 < \cdots < u_k$ that have regularity $-1 \le r_j \le p$ on the point u_j, where regularity -1 means discontinuous functions. Notice that a regularity condition on u_0 or u_k are computed with respect to the null function. If $r_j = -1$ for every j, then $\dim V = k(p+1)$. In general, each regularity decreases the dimension, so

$$\dim V = k(p+1) - \sum_{j=0}^{k}(r_j+1)$$

Choose the knots u_i with multiplicity $s_i = p - r_i$, so that the degree p B-splines have regularity r_i on u_i. The number of such B-splines is

$$\sum_{i=0}^{k} s_i - (p+1) = p(k+1) - (p+1) - \sum_{i=0}^{k} r_i = pk - 1 + k + 1 - \sum_{j=0}^{k}(r_j+1) = k(p+1) - \sum_{j=0}^{k}(r_j+1) = \dim V$$

and they all belong to V. We only need to prove they are linearly independent.

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Considering the nodes

$$u_0 = \cdots = u_p \le u_{p+1} \le \cdots \le u_{n+p-1} \le u_{n+p} = \cdots = u_{n+2p}$$

we obtain n functions $N_{0,p}, \dots, N_{n-1,p}$.

Definition 8. The **Dual Polynomials** of $N_{i,p}$ are

$$\psi_{i,p}(y) = (y - u_{i+1})(y - u_{i+2}) \dots (y - u_{i+p})$$

where $\psi_{i,0}(y) = 1$.

For example, given nodes $\{0, 0, 1, 2, 3, 4, 4\}$ on $[0, 4]$, we have that $\psi_{i,1}(y) = y - i$.
An other example, given nodes $\{0, 0, 0, 1, 1, 1\}$ we have $N_{\cdot,2} = (1-u)^2, 2u(1-u), u^2$ and $\psi_{\cdot,2} = y^2, y(y-1), (y-1)^2$.

Theorem 3. *The following recurrence relations hold:*

1. $(y - x)\psi_{i,p-1}(y) = \frac{x-u_i}{u_{i+p}-u_i}\psi_{i,p}(y) + \frac{u_{i+p}-x}{u_{i+p}-u_i}\psi_{i-1,p}(y)$
2. $\psi_{i,p-1}(y) = \frac{1}{u_{i+p}-u_i}\psi_{i-1,p}(y) + \frac{1}{u_{i+p}-u_i}\psi_{i,p}(y)$

Proof. **1)** Fix $y \in \mathbb{R}$ and define $l_y(x) = y - x$. If we interpolate $(u_i, l_y(u_i))$ and $(u_{i+p}, l_y(u_{i+p}))$, obtaining

$$l_y(x) = \frac{x - u_i}{u_{i+p} - u_i} l_y(u_i) + \frac{u_{i+p} - x}{u_{i+p} - u_i} l_y(u_{i+p}).$$

Multiplying $\psi_{i,p-1}(y)$ we get

$$\begin{aligned}(y - x)\psi_{i,p-1}(y) &= \frac{x - u_i}{u_{i+p} - u_i} l_y(u_i)\psi_{i,p-1}(y) + \frac{u_{i+p} - x}{u_{i+p} - u_i} l_y(u_{i+p})\psi_{i,p-1}(y) \\ &= \frac{x - u_i}{u_{i+p} - u_i}\psi_{i-1,p}(y) + \frac{u_{i+p} - x}{u_{i+p} - u_i}\psi_{i,p}(y).\end{aligned}$$

2) Let us derive 1) with respect to x.

$$-\psi_{i,p-1}(y) = \frac{1}{u_{i+p} - u_i}\psi_{i,p}(y) - \frac{1}{u_{i+p} - u_i}\psi_{i-1,p}(y)$$

□

Theorem 4 (local Marsden identity)**.** *Given the nodes*

$$u_0 = \dots = u_p \le u_{p+1} \le \dots \le u_{n+p-1} \le u_{n+p} = \dots = u_{n+2p}$$

and m such that $u_m \neq u_{m+1}$ then for every $x \in [u_m, u_{m+1})$ and $y \in \mathbb{R}$ we have

$$(y - x)^p = \sum_{i=m-p}^{m} N_{i,p}(x)\psi_{i,p}(y).$$

Proof. By induction on p, if $p = 0$, then

$$(y - x)^0 = 1 = N_{m,0}(x)\psi_{m,0}(y).$$

Given $p + 1$, we have

$$\begin{aligned}(y - x)^{p+1} &= (y - x)\sum_{i=m-p}^{m} N_{i,p}(x)\psi_{i,p}(y) \\ &= \sum_{i=m-p}^{m} N_{i,p}(x)\frac{x - u_i}{u_{i+p+1} - u_i}\psi_{i,p+1}(y) + N_{i,p}(x)\frac{u_{i+p+1} - x}{u_{i+p+1} - u_i}\psi_{i-1,p+1}(y) \\ &= \sum_{i=m-p-1}^{m} \frac{x - u_i}{u_{i+p+1} - u_i}\psi_{i,p+1}(y)N_{i,p}(x) + \sum_{i=m-p}^{m+1} \frac{u_{i+p+1} - x}{u_{i+p+1} - u_i}\psi_{i-1,p+1}(y)N_{i,p}(x) \\ &= \sum_{i=m-p-1}^{m} \frac{x - u_i}{u_{i+p+1} - u_i}\psi_{i,p+1}(y)N_{i,p}(x) + \frac{u_{i+p+2} - x}{u_{i+p+2} - u_{i+1}}\psi_{i,p+1}(y)N_{i+1,p}(x) \\ &= \sum_{i=m-p-1}^{m} \psi_{i,p+1}(y)N_{i,p+1}(x)\end{aligned}$$

□

If we compute the $p-k$ th derivative with respect to y of the Marsden equality, we obtain

$$\frac{p!}{k!}(y-x)^k = p(p-1)\dots(k+1)(y-x)^k = \sum_{i=m-p}^{m} N_{i,p}(x)D^{p-k}\psi_{i,p}(y)$$

$$\implies (y-x)^k = \sum_{i=m-p}^{m} \frac{k!}{p!}N_{i,p}(x)D^{p-k}\psi_{i,p}(y).$$

If $k=0$, we have

$$1 = \sum_{i=m-p}^{m} \frac{1}{p!}N_{i,p}(x)D^{p}\psi_{i,p}(y) = \sum_{i=m-p}^{m} N_{i,p}(x)$$

since $\psi_{i,p}(y)$ is a monic polynomial of degree p, so we have proven that the B-splines are a partition of unit. When we substitute $y=0$

$$x^k = \sum_{i=m-p}^{m} (-1)^k\frac{k!}{p!}N_{i,p}(x)D^{p-k}\psi_{i,p}(0).$$

can be seen that every monomial $x^0,\dots,x^p$ is generated by $N_{0,p}(x),\dots,N_{p,p}(x)$. They are thus linearly independent on $[u_m,u_{m+1})$, or we can also say that they form a local basis.

Moving on the global case, consider the nodes

$$a = u_0 = \dots = u_p \le u_{p+1} \le \dots \le u_{n+p-1} \le u_{n+p} = \dots = u_{n+2p} = b.$$

Definition 9. The nodes vector is said to be $(p+1)-$**Regular** when $u_j < u_{j+p+1}$ for every index j.

A $(p+1)$ regular vector nodes is necessary to have all B-splines $N_{j,p}$ not null.

Theorem 5. *If the nodes vector id $(p+1)-$regular, then $N_{0,p}(x),\dots,N_{p,p}(x)$ are linearly independent on $[a,b]$*

Proof. Suppose $s(x)=\sum c_jN_{j,p}(x)$ is identically zero. Fix j and find $j\le m_j\le j+p$ such that $u_{m_j}<u_{m_j+1}$. Using the consequences of Marsden identity, we know that $c_{m_j}=\dots=c_{m_j-p}=0$ and in particular $c_j=0$. □

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Exercise 2. *Given $P_0,\dots,P_n$ control points on nodes $u_0=u_1=u_2<u_3<\dots<u_n<u_{n+1}=u_{n+2}=u_{n+3}$, prove that the quadratic B-spline curve $\mathscr{C}(u)=\sum_{i=0}^n N_{i,2}(u)P_i$ is tangent to every segment P_jP_{j+1}.*

We already know that $\mathscr{C}(u_0)=P_0$ and $\mathscr{C}(u_{n+3})=P_n$ and that $\mathscr{C}$ is tangent to P_0P_1 and $P_{n-1}P_n$. If we consider a fixed index j,

$$\mathscr{C}(u_j) = N_{j-2,2}(u_j)P_{j-2} + N_{j-1,2}(u_j)P_{j-1}$$

that is a convex combination of P_{j-2} and P_{j-1} since $N_{\cdot,2}$ are a partition of the unity (if the knots are equidistant, then it is the middle point of the segment $P_{j-2}P_{j-1}$). The same holds for derivatives, since quadratic B-splines are in C^1.

$$\mathscr{C}'(u_j) = N'_{j-2,2}(u_j)P_{j-2} + N'_{j-1,2}(u_j)P_{j-1} = N_{j-1,1}(u_j)Q_{j-2} = 2N_{j-1,1}(u_j)\frac{P_{j-1}-P_{j-2}}{u_{i+1}-u_{i-1}}$$

so the curve cross the segment $P_{j-2}P_{j-1}$ and it is tangent to the segment since the derivative is parallel to the segment.

4 Rational Bezier Curve

Given a degree n, a set of control points $\boldsymbol{P}=\{P_0,P_1,\dots,P_n\in\mathbb{R}^d\}$ and some real valued positive weights $\boldsymbol{w}=\{w_0,\dots,w_n\}$, we can define

Definition 10. The **Rational Bezier Curve** associated to $\boldsymbol{P}, \boldsymbol{w}$ is defined as

$$\mathscr{C}(t) = \frac{\sum_{i=0}^n B_i^{(n)}(t) w_i P_i}{\sum_{i=0}^n B_i^{(n)}(t) w_i} = \sum_{i=0}^n R_{i,n}(t) P_i$$

where $B_i^{(n)}$ are Bernstein polynomials on $[0,1]$.

In the definition, $R_{i,n}$ are called *basis rational functions* and

$$R_{j,n}(t) = \frac{B_j^{(n)}(t) w_j}{\sum_{i=0}^n B_i^{(n)}(t) w_i}.$$

One can see that

- $R_{j,n}(t)$ are nonnegative functions
- $R_{j,n}(t)$ are a partition of unity
- $R_{0,n}(0) = R_{n,n}(1) = 1$
- $R_{j,n}(0) = 0 \quad \forall j \neq 0, \qquad R_{j,n}(1) = 0 \quad \forall j \neq n$
- If all weights are equal, $R_{j,n}(t) = B_j^{(n)}(t)$
- If we multiply all weights by a common factor, the functions $R_{j,n}(t)$ do not change

The property of convex envelope of the curve is preserved, altogether with the affine invariance. Moreover, $\mathscr{C}(0) = P_0$ and $\mathscr{C}(1) = P_n$ and the variation diminishing property still holds, since the collocation matrices of $R_{i,n}$ are still Totally Nonnegative.

Lemma 4. *The collocation matrices of $R_{i,n}$ are Totally Nonnegative.*

Proof. Given $t_1 < \cdots < t_r$ collocation points in $[0,1]$, we have

$$A = \begin{pmatrix} R_{0,n}(t_1) & \dots & R_{n,n}(t_1) \\ \vdots & & \vdots \\ R_{0,n}(t_r) & \dots & R_{n,n}(t_r) \end{pmatrix}$$

where every entry of a fixed row has the same denominator, so we can factorize

$$A = \begin{pmatrix} w(t_1) & & \\ & \ddots & \\ & & w(t_r) \end{pmatrix} \begin{pmatrix} B_0^{(n)}(t_1) & \dots & B_n^{(n)}(t_1) \\ \vdots & & \vdots \\ B_0^{(n)}(t_r) & \dots & B_n^{(n)}(t_r) \end{pmatrix} \begin{pmatrix} w_0 & & \\ & \ddots & \\ & & w_n \end{pmatrix}$$

where $w^{-1}(t) = \sum_{i=0}^n B_i^{(n)}(t) w_i$. All the matrices in the decomposition are TN, so the product A is TN. □

For example, let us consider a quarter of circumference

$$\mathscr{C} = \{ (x,y) \mid x^2 + y^2 = 1, x \geq 0, y \geq 0 \}.$$

How can we find control points and weight to obtain $\mathscr{C}$? The equation for the curve is

$$x(t) = \frac{1-t^2}{1+t^2} \qquad y(t) = \frac{2t}{1+t^2}$$

so we can rearrange the formula and obtain

$$\mathscr{C}(t) = \frac{(1-t)^2\binom{1}{0} + 2t(1-t)\binom{1}{1} + 2t^2\binom{0}{1}}{(1-t)^2 + 2t(1-t) + 2t^2} \qquad w_0 = w_1 = 1, w_2 = 2 \quad P_0 = \begin{pmatrix}1\\0\end{pmatrix}, P_1 = \begin{pmatrix}1\\1\end{pmatrix}, P_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

For the general quadratic setting with 3 control points we have

$$\mathscr{C}(t) = \frac{(1-t)^2 w_0 P_0 + 2t(1-t)w_1 P_1 + t^2 w_2 P_2}{(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2}$$

so we can classify the typology of the resulting conic by analysing the behaviour at infinite. Focusing on the denominator, we have

$$(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2 = 0 \to t^2(w_0 - 2w_1 + w_2) + 2t(w_1 - w_0) + w_0 = 0 \to \Delta/4 = w_1^2 - w_0 w_2$$

and if we call $k = w_0 w_2 / w_1^2$, then the conic is an ellipsis when $k > 1$ (no point at infinite), a parabola if $k = 1$ (1 point at infinite) and an hyperbole when $k < 1$ (2 points at infinite).

We can continue by considering the projective space $\mathbb{P}(\mathbb{R}^4)$ and the embedded space $\mathbb{R}^3$. Using the usual coordinates, and given a $w \neq 0$, we have the embedding $E^w : (x, y, z) \hookrightarrow (xw, yw, zw, w)$ from $\mathbb{R}^3$ to $\mathbb{R}^4$ and the projection $H : (x, y, z, w) \to (x/w, y/w, z/w)$ from $\mathbb{R}^4$ to $\mathbb{R}^3$.

Given $P_0, \dots, P_n$ control points in $\mathbb{R}^3$, where $P_i = (x_i, y_i, z_i)$, and weights $w_0, \dots, w_n$, we call $P_i^w = E^{w_i}(P_i) = (w_i x_i, w_i y_i, w_i z_i, w_i)$ and define a classic Bezier curve in $\mathbb{R}_4$

$$\mathscr{C}^w(t) = \sum_{i=0}^{n} B_i^{(n)}(t) P_i^w = \sum_{i=0}^{n} B_i^{(n)}(t) w_i \begin{pmatrix} P_i \\ 1 \end{pmatrix}$$

When we apply H, we find the rational Bezier curve in $\mathbb{R}^3$. As a consequence, Bezier rational curves on $\mathbb{R}^d$ are just Bezier curves on $\mathbb{P}(\mathbb{R}^d)$, or also the projection of Bezier curves on $\mathbb{R}^{d+1}$.

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What's the behaviour of a rational Bezier curve when a projective transformation is applied? Remember that an affine map on the plane is described by

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 1 & a_1 2 & t_1 \\ a_2 1 & a_2 2 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

A projective transformation on the plane is the projection of a linear transformation on $\mathbb{R}^3$

$$P = \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \to A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \to H\left(A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right) = [A]P.$$

When we apply this transformation to a Bezier curve we obtain

$$H(A\mathscr{C}^w(t)) = H\left(A \sum_{i=0}^{n} B_i^{(n)}(t) P_i^w \right) = H\left(\sum_{i=0}^{n} B_i^{(n)}(t) w_i A \begin{pmatrix} P_i \\ 1 \end{pmatrix} \right) = H\left(\sum_{i=0}^{n} B_i^{(n)}(t) w_i \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} \right)$$

$$= \begin{pmatrix} \frac{\sum_{i=0}^{n} B_i^{(n)}(t) w_i x_i'}{\sum_{i=0}^{n} B_i^{(n)}(t) w_i z_i'} \\ \frac{\sum_{i=0}^{n} B_i^{(n)}(t) w_i y_i'}{\sum_{i=0}^{n} B_i^{(n)}(t) w_i z_i'} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i') \frac{x_i'}{z_i'}}{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')} \\ \frac{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i') \frac{y_i'}{z_i'}}{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')([A]P_i)_x}{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')} \\ \frac{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')([A]P_i)_y}{\sum_{i=0}^{n} B_i^{(n)}(t)(w_i z_i')} \end{pmatrix}$$

so the result is an other Bezier curve with weights $w_i z_i'$ and control points $[A]P_i$. Every algorithm $\mathscr{A}$ designed on polynomial Bezier curves can be applied in the rational case through projection/homogenization.

$$\mathscr{C} \to \mathscr{C}^w \xrightarrow{\mathscr{A}} \mathscr{A}(\mathscr{C}^w) \xrightarrow{H} H(\mathscr{A}(\mathscr{C}^w))$$

For example, we can apply de Casteljoe algorithm.

Given P_i control points, and $t \in [0, 1]$ how o we compute $\mathscr{C}(t)$?

First Method) Compute $\hat{P}_i = \binom{P_i}{1}$, and recursively $\hat{P}_i^{(r)} = \hat{P}_i^{(r-1)}(1-t) + \hat{P}_{i+1}^{(r-1)} t$. In the end, project to obtain the answer $\mathscr{C}(t) = H(\hat{P}_0^{(n)})$.

Second Method)

$$C_i^{(r)}(t) = \frac{(1-t) C_i^{(r-1)} w_i^{(r-1)} + t C_{i+1}^{(r-1)} w_{i+1}^{(r-1)}}{(1-t) w_i^{(r-1)} + t w_i^{(r-1)}}, \qquad w_i^{(r)}(t) = (1-t) w_i^{(r-1)}(t) + t w_{i+1}^{(r-1)}(t), \qquad \mathscr{C}(t) = C_0^{(n)}(t)$$

4.1 Re-parametrization

Given a function $f : [c, d] \to [a, b]$ such that $f \in C^1$ and $f'(s) > 0$ for every s, and moreover $f(c) = a$, $f(d) = b$, we can reparametrize a curve $\mathscr{C} : [a, b] \to \mathbb{R}^n$ through f

$$\mathscr{C}_f(s) = \mathscr{C}(f(s)).$$

If we consider $a < u_1 < \cdots < u_n < b$, then we may want a curve that reaches the points $\mathscr{C}(u_i)$ in correspondence to other nodes $c < s_1 < \cdots < s_n < d$. In this case, one can find f that maps $u_i \to s_i$.

An other application of reparametrization is to change the derivatives

$$\mathscr{C}_f'(s) = \mathscr{C}'(f(s))f'(s)$$

or to change the the weights. For example, we can always bring a curve to its *Standard Form*, meaning that the first and last weights are both unitary.

Lemma 5. *Given $\mathscr{C}(t)$ and $\widetilde{\mathscr{C}}(t)$ rational Bezier curves with the same degree, same control points and weights w_i and $\widetilde{w}_i$ respectively, with $\widetilde{w}_i = \omega^i w_i$, then $\mathscr{C}$ and $\widetilde{\mathscr{C}}$ have the same support.*

Proof. Suppose we want to reparametrize the curve through a rational projective function

$$f(s) = \frac{as + b}{cs + 1}.$$

We want $f : [0, 1] \to [0, 1]$, so $f(0) = 0 \to b = 0$ and $f(1) = 1 \to a = c + 1$, so

$$f(s) = \frac{as}{(a - 1)s + 1}$$

and $sign(f'(s)) = sign(a)$, so $a > 0$. If we compute $\mathscr{C}(f(s))$, we find a curve with the same control points, same degree, but weights $a^i w_i$. □

Remember that $w_i > 0$, so $\omega > 0$.

Corollary 2. *Given a rational Bezier curve $\mathscr{C}$ with weights w_i, we can define $\widetilde{w}_i = \omega^i w_i / w_0$, $\omega = \sqrt[n]{w_0 / w_n}$, and obtain a curve $\widetilde{\mathscr{C}}$ with the same support and in standard form.*

Remember that a Bezier curve does not change when we multiply all weights by a common factor.

Corollary 3. *Given a degree 2 rational Bezier curve, it is determined by the control points, and only 1 weight w, since it is equivalent to its standard form. When $w < 1$, it is an ellipsis, $w = 1$ a parabola, $w > 1$ an hyperbole.*

Notice that not all conic curves can be represented by a Bezier curve. For example, half of a circumference is not a Bezier curve, since the tangent lines at the extreme points do not meet, so we can not define the middle control point.

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Exercise 3. *Represent the circumference as a NURBS curve with 9 points.*

The idea is to glue quarters of circumference, since we know they are already NURBS curves. We thus take

$$U = \left\{0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1\right\}, \qquad w = \left\{1, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1\right\},$$

$$P_1 = (1, 1), \quad P_2 = (0, 1), \quad P_3 = (-1, 1), \quad P_4 = (-1, 0)$$
$$P_5 = (-1, -1), \quad P_6 = (0, -1), \quad P_7 = (1, -1), \quad P_8 = P_0 = (1, 0).$$

In this case we have

$$\mathscr{C}^w(u) = \sum_{i=0}^{8} N_{i,2}(u) P_i^w, \qquad \mathscr{C}'^w(u) = \sum_{i=0}^{7} N_{i+1,1}(u) Q_i^w, \quad Q_i^w = 2\frac{P_{i+1}^w - P_i^w}{u_{i+3} - u_{i+1}}$$

If $u \in [0, 1/4]$, then we have only two non zero $N_{i,1}$, so

$$\mathscr{C}'^w(u) = (1-4u)2\frac{P_1^w - P_0^w}{u_3 - u_1} + 8u\frac{P_2^w - P_1^w}{u_4 - u_2} = 8(1-4u)(P_1^w - P_0^w) + 32u(P_2^w - P_1^w)$$

$$\implies \lim_{u \to \frac{1}{4}^-} \mathscr{C}'^w(u) = 8(P_2^w - P_1^w).$$

If $u \in [0, 1/2]$, then

$$\mathscr{C}'^w(u) = (2-4u)2\frac{P_3^w - P_2^w}{u_5 - u_3} + (4u-1)2\frac{P_4^w - P_3^w}{u_6 - u_4} = 8(2-4u)(P_3^w - P_2^w) + 8(4u-1)(P_4^w - P_3^w)$$

$$\implies \lim_{u \to \frac{1}{4}^+} \mathscr{C}'^w(u) = 8(P_3^w - P_2^w).$$

We have that the two quantities are different, because on the last coordinate we find

$$[8(P_2^w - P_1^w)]_3 = 8(w_2 - w_1) = -8(w_3 - w_2) = -[8(P_3^w - P_2^w)]_3$$

so the curve is not C^1. If we consider the rational projected curve, and call $\mathscr{A}(u)$ the first two coordinates, then

$$\mathscr{C}(u) = \frac{\mathscr{A}(u)}{w(u)} \implies \mathscr{C}'(u) = \frac{\mathscr{A}'(u) - w'(u)\mathscr{C}(u)}{w(u)}$$

$$\implies \lim_{u \to \frac{1}{4}^-} \mathscr{A}'(u) = 8(w_2 P_2 - w_1 P_1), \qquad \lim_{u \to \frac{1}{4}^+} \mathscr{A}'(u) = 8(w_3 P_3 - w_2 P_2), \qquad (1/4) = P_2, \quad w(1/4) = 1$$

$$\implies \lim_{u \to \frac{1}{4}^-} \mathscr{C}'(u) = \frac{8(w_2 P_2 - w_1 P_1) - 8(w_2 - w_1) P_2}{1} = 8w_1(P_2 - P_1) = 4\sqrt{2}(-1, 0),$$

$$\implies \lim_{u \to \frac{1}{4}^+} \mathscr{C}'(u) = \frac{8(w_3 P_3 - w_2 P_2) - 8(w_3 - w_2) P_2}{1} = 8w_3(P_3 - P_2) = 4\sqrt{2}(-1, 0)$$

and we can verify this property for every node, so the rational curve is a C^1 curve.

Half Circumference We know a rational Bezier parametrization of half circumference, using the point at infinite.

$$P_0^w = (1, 0, 1), \quad P_1^w = (0, 1, 0), \quad P_2^w = (-1, 0, 1), \quad U = \{\, 0, 0, 0, 1, 1, 1 \,\}.$$

If we want to insert a new node

$$\overline{U} = \{\, 0, 0, 0, 1/2, 1, 1, 1 \,\}$$

we need a new control points

$$Q_i^w = (1-\alpha_i) P_i^w + \alpha_i P_{i+1}^w, \qquad \alpha_i = \begin{cases} 1 & i \le k-p \\ \frac{\overline{u} - u_i}{u_{i+p} - u_i} & k-p < i \le k \\ 0 & i > k \end{cases} \implies \alpha = (0, 1/2, 1/2, 1)$$

$$\implies Q_0^w = P_0^w, \quad Q_1^w = \frac{P_0^w + P_1^w}{2} = \frac{1}{2}(1, 1, 1), \quad Q_2^w = \frac{P_1^w + P_2^w}{2} = \frac{1}{2}(-1, 1, 1), \quad Q_3^w = P_2^w.$$

The projection is a degree 2 rational NURBS curve with 4 control points in the plane, without infinite points. The control points are

$$Q_0 = (1, 0), \quad Q_1 = (1, 1), \quad Q_2 = (-1, 1), \quad Q_3 = (-1, 0)$$

with weights $(0, 1/2, 1/2, 1)$.

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5 Bezier Surfaces

Given u, v two variables on $[0,1]$, control points $P_{i,j} \in \mathbb{R}^N$ with $N \geq 3$, and basis functions $\{ f_1, \dots, f_n \}$ and $\{ g_1, \dots, g_m \}$ a *Bezier surface* is defined as

$$\mathscr{S}(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(u) g_j(v) P_{i,j}.$$

They have all the good properties of Bezier curves, except for the diminishing variation property, that is not well defined in this case.

If we have also weights $w_{i,j}$, we can define *rational Bezier surfaces* as

$$\mathscr{S}(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}(u,v) P_{i,j}$$

where

$$R_{i,j} = \frac{f_i(u) g_j(v) w_{i,j}}{\sum_{k=1}^{n} \sum_{r=1}^{m} f_k(u) g_r(v) w_{k,r}}.$$

Notice that if all the weights are equal, then we obtain again the classical Bezier surfaces.

Exercise 4. *Let $\mathscr{S}(u,v)$ be the bi-quadratic Bezier surface with control points*

$$P_{0,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,1} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,2} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \quad P_{1,0} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad P_{1,1} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$$P_{1,2} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \quad P_{2,0} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \quad P_{2,1} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}, \quad P_{2,2} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

What is $\mathscr{S}(1/2, 1/2)$?

We can us the De Casteljoe algorithm. Apply the algorithm first on the rows:

$$P_{0,0}^{1,0} = \frac{1}{2} P_{0,0}^{0,0} + \frac{1}{2} P_{0,1}^{0,0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,1}^{1,0} = \frac{1}{2} P_{0,1}^{0,0} + \frac{1}{2} P_{0,2}^{0,0} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,0}^{2,0} = \frac{1}{2} P_{0,0}^{1,0} + \frac{1}{2} P_{0,1}^{1,0} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

$$P_{1,0}^{1,0} = \frac{1}{2} P_{1,0}^{0,0} + \frac{1}{2} P_{1,1}^{0,0} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad P_{1,1}^{1,0} = \frac{1}{2} P_{1,1}^{0,0} + \frac{1}{2} P_{1,2}^{0,0} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad P_{1,0}^{2,0} = \frac{1}{2} P_{1,0}^{1,0} + \frac{1}{2} P_{1,1}^{1,0} = \begin{pmatrix} 2 \\ 2 \\ 0.5 \end{pmatrix}.$$

$$P_{2,0}^{1,0} = \frac{1}{2} P_{2,0}^{0,0} + \frac{1}{2} P_{2,1}^{0,0} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad P_{2,1}^{1,0} = \frac{1}{2} P_{2,1}^{0,0} + \frac{1}{2} P_{2,2}^{0,0} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}, \quad P_{2,0}^{2,0} = \frac{1}{2} P_{2,0}^{1,0} + \frac{1}{2} P_{2,1}^{1,0} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

Then we apply it again on the three results:

$$P_{0,0}^{2,1} = \frac{1}{2} P_{0,0}^{2,0} + \frac{1}{2} P_{1,0}^{2,0} = \begin{pmatrix} 2 \\ 1 \\ 0.25 \end{pmatrix}, \quad P_{1,0}^{2,1} = \frac{1}{2} P_{1,0}^{2,0} + \frac{1}{2} P_{2,0}^{2,0} = \begin{pmatrix} 2 \\ 3 \\ 1.75 \end{pmatrix}, \quad P_{0,0}^{2,2} = \frac{1}{2} P_{0,0}^{2,1} + \frac{1}{2} P_{1,0}^{2,1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

5.1 Matricial Form

Matricial Form of Bezier surface:

$$\mathscr{S}(u,v) = \begin{pmatrix} B_0^{(n)}(u) & B_1^{(n)}(u) & \dots & B_n^{(n)}(u) \end{pmatrix} \begin{pmatrix} P_{0,0} & \dots & P_{0,m} \\ \vdots & & \vdots \\ P_{n,0} & \dots & P_{n,m} \end{pmatrix} \begin{pmatrix} B_0^{(m)}(v) \\ B_1^{(m)}(v) \\ \vdots \\ B_m^{(m)}(v) \end{pmatrix}.$$

If we want a different basis, for example a monomial basis, we have

$$\begin{pmatrix} B_0^{(m)}(v) \\ B_1^{(m)}(v) \\ \vdots \\ B_m^{(m)}(v) \end{pmatrix} = M_m \begin{pmatrix} 1 \\ v \\ \vdots \\ v^m \end{pmatrix} \implies \mathscr{S}(u,v) = \begin{pmatrix} 1 & n & \dots & u^n \end{pmatrix} (M_n)^T \begin{pmatrix} P_{0,0} & \dots & P_{0,m} \\ \vdots & & \vdots \\ P_{n,0} & \dots & P_{n,m} \end{pmatrix} M_m \begin{pmatrix} 1 \\ v \\ \vdots \\ v^m \end{pmatrix}.$$

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6 Interpolation with B-splines

Given $n+1$ points $Q_i \in \mathbb{R}^d$ and a degree $p > 0$, can we find a B-spline curve of degree p that includes the points Q_i?

$$\mathscr{C}(u) = \sum_{i=0}^{n} N_{i,p}(u) P_i$$

and we want to find the right P_i and the right vector of nodes $u_0, \dots, u_m$ with $m = n + p + 1$. Moreover, we must require the existence of points t_k such that $\mathscr{C}(t_k) = Q_k$ for every k.

Suppose that we fixed u_i and t_k. We have the conditions

$$Q_k = \sum_{i=0}^{n} N_{i,p}(t_k) P_i \implies \begin{pmatrix} Q_{0,1} & \dots & Q_{n,1} \\ \vdots & & \vdots \\ Q_{0,d} & \dots & Q_{n,d} \end{pmatrix} = \begin{pmatrix} P_{0,1} & \dots & P_{n,1} \\ \vdots & & \vdots \\ P_{0,d} & \dots & P_{n,d} \end{pmatrix} \begin{pmatrix} N_{0,p}(t_0) & \dots & N_{0,p}(t_n) \\ \vdots & & \vdots \\ N_{n,p}(t_0) & \dots & N_{n,p}(t_n) \end{pmatrix}$$

that is a non singular linear system. There are many ways to choose t_k.

- Equidistant: $t_k = k/n$
- Chord length: $l = \sum_{i=0}^{n} |Q_i - Q_{i-1}| \implies t_k = t_{k-1} + \frac{|Q_k - Q_{k-1}|}{l}, \quad t_0 = 0$
- Centripetal method: $l = \sum_{i=0}^{n} \sqrt{|Q_i - Q_{i-1}|} \implies t_k = t_{k-1} + \frac{\sqrt{|Q_k - Q_{k-1}|}}{l}, \quad t_0 = 0$

If we try to confront the curves obtained with the different choices of t_k, we find that the centripetal method produces a good approximation of the control polygon, even though the equidistant method behaves better when the polygon is close to a straight line. Even the nodes u_i con be chosen in different ways, excluding the obligatory conditions $u_0 = \cdots = u_p = 0$ and $u_{m-p} = \cdots = u_m = 1$.

- Equidistant. The problem is that the linear system tends to be ill-conditioned.
- Averaging: $u_{j+p} = \frac{1}{p} \sum_{k=j}^{j+p-1} t_k$. It leads to a totally nonnegative banded matrix that can be factorized through Gauss into LU without pivoting.

Exercise 5. *Interpolate*

$$Q_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} -4 \\ -3 \end{pmatrix},$$

with cubic B-splines.

$n = 4$, $p = 3$, $m = n + p + 1 = 8$. We use chord length and averaging nodes

$$l = 17, \quad t_0 = 0, \quad t_1 = 5/17, \quad t_2 = 9/17, \quad t_3 = 14/17, \quad t_4 = 1,$$

$$u_0 = u_1 = u_2 = u_3 = 0, \quad u_4 = 28/51, \quad u_5 = u_6 = u_7 = u_8 = 1.$$

$$N = \begin{pmatrix} 1 & N_{0,3}(t_1) & N_{0,3}(t_2) & 0 & 0 \\ 0 & N_{1,3}(t_1) & N_{1,3}(t_2) & N_{1,3}(t_3) & 0 \\ 0 & N_{2,3}(t_1) & N_{2,3}(t_2) & N_{2,3}(t_3) & 0 \\ 0 & N_{3,3}(t_1) & N_{3,3}(t_2) & N_{3,3}(t_3) & 0 \\ 0 & 0 & 0 & N_{4,3}(t_3) & 1 \end{pmatrix}$$

An other method to choice t_k and u_i is called *universal method* since it does not depends on Q_i. We choose u_i equidistant, and then we choose t_k as the maxima of the B-splines functions.

In the previous exercise, we would have $u_4 = 1/2$ and

$$t_0 = 0, \quad t_1 = 1/3, \quad t_2 = 1/2, \quad t_3 = 2/3, \quad t_4 = 1.$$

In this case, the problem is invariant by affine transformation, meaning that if $Q = PN$, then the control point associated to $f(Q)$ are $f(P)$, since

$$f(Q) = MQ + ve^T \implies f(Q) = MPN + ve^T = (MP + ve^T)N = f(P)N$$

where we used that t_k are independent from Q_i (in fact it works even if t_k are equispaced).

6.1 Surfaces

In the case of B-spline surfaces, we can repeat the analysis with $Q_{i,j}$ interpolation points and $P_{i,j}$ control points to find. If we set p, q the degrees and the nodes, then we have also to find $t_{k,h} = (r_k, s_h)$ such that $\mathscr{S}(t_{k,h}) = Q_{k,h}$.

For every column, we take $Q_{0,k}, \dots, Q_{n,k}$ and determine (using some methods for curves) the nodes $r_i^{(k)}$. Eventually, we get the average on k, so that we obtain our candidates r_i. The same argument can be applied to find s_j, so we have our $t_{k,h}$.

$$Q_{h,k} = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(r_h) N_{j,q}(s_k) P_{i,j}, \qquad A_{i,k} = \sum_{j=1}^{m} N_{j,q}(s_k) P_{i,j}$$

$$\implies Q_{h,k} = \sum_{i=1}^{n} N_{i,p}(r_h) A_{i,k}.$$

We can solve first for $A_{i,k}$ and then for $P_{i,j}$ like we did for curves. They are a lot of linear systems, but they can be computed with only 2 factorizations LU.

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7 Triangular Bernstein Surfaces

To define a surface on a triangle, we need to use *Barycentric Coordinates*. In fact, suppose $T \subseteq \mathbb{R}^2$ is a triangle with vertices V_1, V_2, V_3 not on the same line. The coordinates of the vertices will be $V_{i,x}, V_{i,y}$. Given $X \in \mathbb{R}^2$, we can compute its uniquely determined barycentric coordinates $X = (\tau_1, \tau_2, \tau_3)$ w.r.t. T given by the system

$$\begin{cases} X = \tau_1 V_1 + \tau_2 V_2 + \tau_3 V_3, \\ \tau_1 + \tau_2 + \tau_3 = 1. \end{cases}$$

In particular,

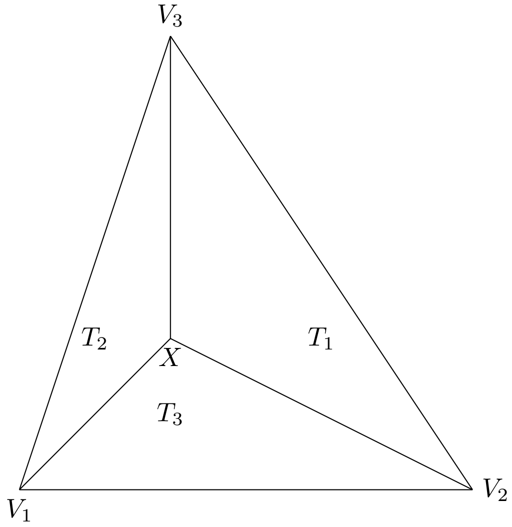
$$V_1 = (1, 0, 0), \quad V_2 = (0, 1, 0), \quad V_3 = (0, 0, 1).$$

They are called barycentric coordinates since the barycentre of the triangle T is $G = (1/3, 1/3, 1/3)$. We can obtain the coordinates solving the equation

$$\begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & V_{2,x} & V_{3,x} \\ V_{1,y} & V_{2,y} & V_{3,y} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} 1 \\ X_x \\ X_y \end{pmatrix}.$$

Using Cramer, we obtain

$$\tau_1 = \frac{\det \begin{pmatrix} 1 & 1 & 1 \\ X_x & V_{2,x} & V_{3,x} \\ Y_y & V_{2,y} & V_{3,y} \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & V_{2,x} & V_{3,x} \\ V_{1,y} & V_{2,y} & V_{3,y} \end{pmatrix}}, \quad \tau_2 = \frac{\det \begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & X_x & V_{3,x} \\ V_{1,y} & X_y & V_{3,y} \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & V_{2,x} & V_{3,x} \\ V_{1,y} & V_{2,y} & V_{3,y} \end{pmatrix}}, \quad \tau_3 = \frac{\det \begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & V_{2,x} & X_x \\ V_{1,y} & V_{2,y} & X_y \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ V_{1,x} & V_{2,x} & V_{3,x} \\ V_{1,y} & V_{2,y} & V_{3,y} \end{pmatrix}},$$



$$\tau_1 = \frac{surf(T_1)}{surf(T)}, \quad \tau_2 = \frac{surf(T_2)}{surf(T)}, \quad \tau_3 = \frac{surf(T_3)}{surf(T)}.$$

Definition 11. Given a triangle T and degree p, the **Bernstein triangular polynomials** are

$$B^{(p)}_{i,j,k}(X) = \frac{p!}{i!j!k!}\tau_1^i \tau_2^j \tau_3^k$$

where $i + j + k = p$.

Notice it is a bivariate polynomial with two indices. They form a basis for the space of bivariate polynomials on T, since the dimension of the space and the number of such polynomials are both $\binom{p+2}{2}$, and they are independent since

$$\sum_{i,j} c_{i,j}\tau_1^i\tau_2^j\tau_3^k = 0 \implies \sum_{i,j} c_{i,j}u^i v^j \tau_3^p = 0, \qquad u = \frac{\tau_1}{\tau_3}, \quad v = \frac{\tau_2}{\tau_3} \implies c_{i,j} = 0$$

For example, for $p = 2$, we have six Bernstein polynomials

$$B^{(2)}_{1,0,1}(X) = 2\tau_1\tau_3, \quad B^{(2)}_{1,1,0}(X) = 2\tau_1\tau_2, \quad B^{(2)}_{0,1,1}(X) = 2\tau_2\tau_3,$$

$$B^{(2)}_{2,0,0}(X) = \tau_1^2, \quad B^{(2)}_{0,2,0}(X) = \tau_2^2, \quad B^{(2)}_{0,0,2}(X) = \tau_3^2$$

Notice that on the sides of the triangle, we obtain the classic Bernstein polynomials. Notice that if $X \in T$

- $B^{(p)}_{i,j,k}(X) \geq 0$,
- $\sum_{i+j+k=p} B^{(p)}_{i,j,k}(X) = (\tau_1 + \tau_2 + \tau_3)^p = 1$,
- $B^{(p)}_{i,j,k}(X) = \tau_1 B^{(p-1)}_{i-1,j,k}(X) + \tau_2 B^{(p-1)}_{i,j-1,k}(X) + \tau_3 B^{(p-1)}_{i,j,k-1}(X)$,
- $B^{(p)}_{i,j,k}(X) = (\tau_1 + \tau_2 + \tau_3)B^{(p)}_{i,j,k}(X) = \frac{i+1}{p+1}B^{(p+1)}_{i+1,j,k}(X) + \frac{j+1}{p+1}B^{(p+1)}_{i,j+1,k}(X) + \frac{k+1}{p+1}B^{(p+1)}_{i,j,k+1}(X)$.

Suppose $A, B \in \mathbb{R}^2$ and $u = A - B$ a 'vector'. Let us assign to u the *directional* barycentric coordinates

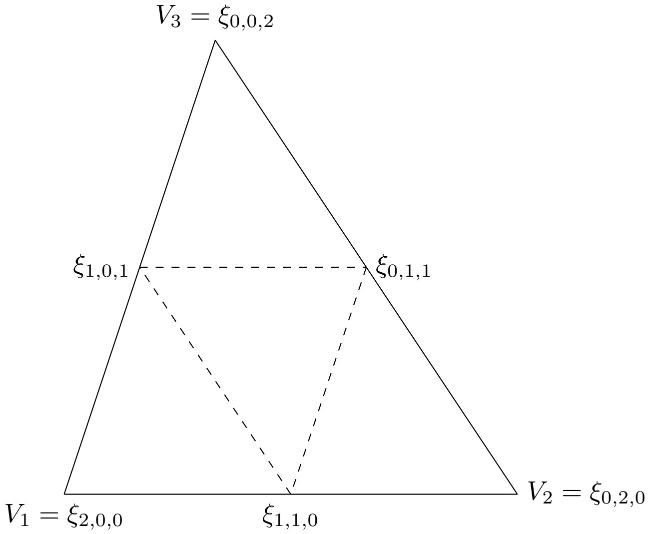
$$u = (\tau_1^A - \tau_1^B, \tau_2^A - \tau_2^B, \tau_3^A - \tau_3^B), \quad \delta_i = \tau_i^A - \tau_i^B, \quad \delta_1 + \delta_2 + \delta_3 = 0.$$

This definition let us continue to write other properties

- $\frac{d}{du}B^{(p)}_{i,j,k}(X) = p\left(\delta_1 B^{(p-1)}_{i-1,j,k}(X) + \delta_2 B^{(p-1)}_{i,j-1,k}(X) + \delta_3 B^{(p-1)}_{i,j,k-1}(X)\right)$

- $\frac{d}{du}\frac{d}{du}B^{(p)}_{i,j,k}(X) = 0 \quad \forall\, u \iff X = \frac{iV_1+jV_2+kV_3}{p} := \xi_{i,j,k}$. It is called Greville(?) Abscissae, and it is the only point of maximum of $B^{(p)}_{i,j,k}$.

For example, if $p = 2$,



and one obtains a splitting into triangles.

7.1 Triangular Bezier Patch

Given a triangle T, a degree p and $P_{i,j,k} \in \mathbb{R}^3$ control points, where $i + j + k = p$, define the Bezier surface

$$\mathscr{S}(X) = \sum_{i+j+k=p} P_{i,j,k} B^{(p)}_{i,j,k}(X). = \sum_{i+j+k=p} P_{i,j,k}\frac{p!}{i!j!k!}\tau_1^i\tau_2^j\tau_3^k$$

It will interpolate the vertices of the triangle T with the points $P_{0,0,p}, P_{0,p,0}, P_{p,0,0}$, and the surface will be contained in the convex envelop of the control points.

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Using the recurrence relations between the Bernstein polynomial, we can derive a De Casteljoe triangular algorithm : if we call $P^p_{i,j,k} = P_{i,j,k}$, then

$$\mathscr{S}(X) = \sum_{i+j+k=p} P_{i,j,k}(\tau_1 B^{(p-1)}_{i-1,j,k}(X) + \tau_2 B^{(p-1)}_{i,j-1,k}(X) + \tau_3 B^{(p-1)}_{i,j,k-1}(X)) = \sum_{i+j+k=p-1} P^{p-1}_{i,j,k} B^{(p-1)}_{i,j,k}(X)$$

where

$$P^{p-1}_{i,j,k} = \tau_1 P^p_{i+1,j,k} + \tau_2 P^p_{i,j+1,k} + \tau_3 P^p_{i,j,k+1}.$$

We can iterate until we arrive to P^1 or P^0. For example, let

$$P_{2,0,0} = \begin{pmatrix}6\\0\\9\end{pmatrix}, \quad P_{0,2,0} = \begin{pmatrix}0\\6\\0\end{pmatrix}, \quad P_{0,0,2} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad P_{1,1,0} = \begin{pmatrix}3\\3\\6\end{pmatrix}, \quad P_{1,0,1} = \begin{pmatrix}3\\0\\0\end{pmatrix} \quad P_{0,1,1} = \begin{pmatrix}0\\3\\0\end{pmatrix}$$

If we want to evaluate $\mathscr{S}(1/3, 1/3, 1/3)$, then

$$P^1_{1,0,0} = \frac{1}{3}(P_{2,0,0} + P_{1,1,0} + P_{1,0,1}) = \begin{pmatrix}4\\1\\5\end{pmatrix},$$

$$P^1_{0,1,0} = \frac{1}{3}(P_{0,2,0} + P_{1,1,0} + P_{0,1,1}) = \begin{pmatrix}1\\4\\2\end{pmatrix},$$

$$P^1_{0,0,1} = \frac{1}{3}(P_{0,0,2} + P_{1,0,1} + P_{0,1,1}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathscr{S}(1/3, 1/3, 1/3) = P^0_{0,0,0} = \frac{1}{3}(P^1_{1,0,0} + P^1_{0,1,0} + P^1_{0,0,1}) = \begin{pmatrix} 2 \\ 2 \\ 7/3 \end{pmatrix}.$$

Notice that we have also a recurrence between Bernstein polynomial that increases the degree of the polynomials.

$$\mathscr{S}(X) = \sum_{i+j+k=p+1} \widehat{P}_{i,j,k} B^{(p+1)}_{i,j,k}(X) \qquad \widehat{P}_{i,j,k} = \frac{i}{p+1} P_{i-1,j,k} + \frac{j}{p+1} P_{i,j-1,k} + \frac{k}{p+1} P_{i,j,k-1}.$$

Moreover, using the formula to derive Bernstein polynomials, we obtain the directional derivatives of the surface wrt $v = (\delta_1, \delta_2, \delta_3)$.

$$D_v \mathscr{S}(X) = p \sum_{i+j+k=p-1} \widetilde{P}_{i,j,k} B^{(p-1)}_{i,j,k}(X) \qquad \widetilde{P}_{i,j,k} = \delta_1 P_{i+1,j,k} + \delta_2 P_{i,j+1,k} + \delta_3 P_{i,j,k+1}.$$

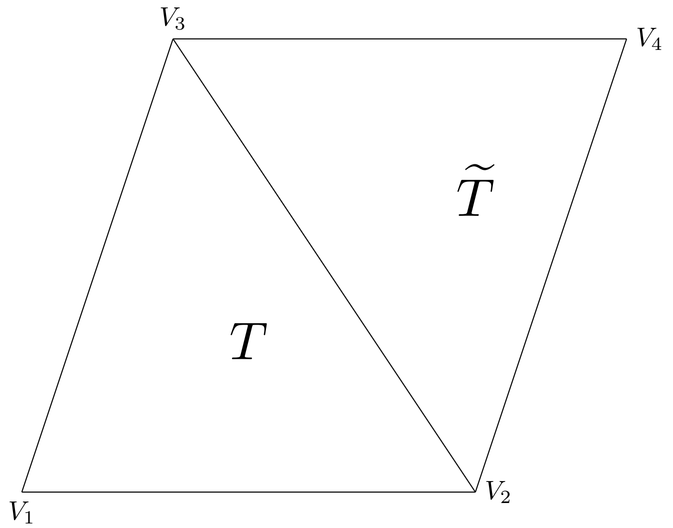
If we compute them at the vertices, for example $V = (1, 0, 0)$, then

$$D_v \mathscr{S}(V) = p \sum_{i+j+k=p-1} (\delta_1 P_{i+1,j,k} + \delta_2 P_{i,j+1,k} + \delta_3 P_{i,j,k+1}) B^{(p-1)}_{i,j,k}(V) = p(\delta_1 P_{p,0,0} + \delta_2 P_{p-1,1,0} + \delta_3 P_{p-1,0,1})$$

hence $\mathscr{S}$ is tangent to the convex hull of the control points on the vertices.

7.2 Gluing patches

Take two triangles $T, \widetilde{T}$ with a common edge. $T = V_1V_2V_3, \widetilde{T} = V_1V_2V_4$, and the barycentric coordinates are (x_1, x_2, x_3) and $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_4)$.



We consider two surfaces

$$\mathscr{S}(X) = \sum_{i+j+k=p} P_{i,j,k} B^{(p)}_{i,j,k}(X), \qquad \widetilde{\mathscr{S}}(X) = \sum_{i+j+k=p} \widetilde{P}_{i,j,k} B^{(p)}_{i,j,k}(X)$$

respectively on T and $\widetilde{T}$. Given $T, \widetilde{T}$ and $P_{i,j,k}$, how do we choose $\widetilde{P}_{i,j,k}$ in order to achieve some regularity on the interface of the patches? Let $X \in V_1V_2$, meaning that

$$X = (t, 1-t, 0) = (t, 1-t, 0)$$

for both coordinate systems.

$$\mathscr{S}(X) = \sum_{i+j=p} P_{i,j,0} \frac{p!}{i!j!} t^i (1-t)^j = \sum_{i+j=p} \widetilde{P}_{i,j,0} \frac{p!}{i!j!} t^i (1-t)^j = \widetilde{\mathscr{S}}(X)$$

for every t, so $P_{i,j,0} = \widetilde{P}_{i,j,0}$. This condition is sufficient and necessary to obtain a continuous function

$$\mathscr{Z}(X) = \begin{cases} \mathscr{S}(X) & X \in T, \\ \widetilde{\mathscr{S}}(X) & X \in \widetilde{T}. \end{cases}$$

To ensure that $\mathscr{Z}$ is also C^1, we need that all diretional derivatives are equal.

$$D_v\mathscr{S}(X) = p \sum_{i+j=p-1} (\delta_1 P_{i+1,j,0} + \delta_2 P_{i,j+1,0} + \delta_3 P_{i,j,1}) \frac{(p-1)!}{i!j!} t^i (1-t)^j$$

$$D_v\widetilde{\mathscr{S}}(X) = p \sum_{i+j=p-1} (\widetilde{\delta}_1 \widetilde{P}_{i+1,j,0} + \widetilde{\delta}_2 \widetilde{P}_{i,j+1,0} + \widetilde{\delta}_4 \widetilde{P}_{i,j,1}) \frac{(p-1)!}{i!j!} t^i (1-t)^j$$

$$\implies \sum_{i+j=p-1} \left(\frac{\delta_1 - \widetilde{\delta}_1}{\widetilde{\delta}_4} P_{i+1,j,0} + \frac{\delta_2 - \widetilde{\delta}_2}{\widetilde{\delta}_4} P_{i,j+1,0} + \frac{\delta_3}{\widetilde{\delta}_4} P_{i,j,1} \right) \frac{1}{i!j!} t^i (1-t)^j = \sum_{i+j=p-1} \widetilde{P}_{i,j,1} \frac{1}{i!j!} t^i (1-t)^j$$

$$\implies \frac{\delta_1 - \widetilde{\delta}_1}{\widetilde{\delta}_4} P_{i+1,j,0} + \frac{\delta_2 - \widetilde{\delta}_2}{\widetilde{\delta}_4} P_{i,j+1,0} + \frac{\delta_3}{\widetilde{\delta}_4} P_{i,j,1} = \widetilde{P}_{i,j,1}$$

where the coefficient are independent from v, since

$$v = \delta_1 V_1 + \delta_2 V_2 + \delta_3 V_3 = \widetilde{\delta}_1 V_1 + \widetilde{\delta}_2 V_2 + \widetilde{\delta}_4 V_4 \implies V_4 = \frac{\delta_1 - \widetilde{\delta}_1}{\widetilde{\delta}_4} V_1 + \frac{\delta_2 - \widetilde{\delta}_2}{\widetilde{\delta}_4} V_2 + \frac{\delta_3}{\widetilde{\delta}_4} V_3.$$

We can thus expect that a regularity of order k require to set the points $\widetilde{P}_{i,j,s}$ for every $s \leq k$.

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8 Subdivision Techniques

To build a curve or surface, we may start from a "control" polygon and we approximate the wanted curve/surface by refining the polygon by adding points. In the field of animation, subdivision techniques on surfaces are widely used, but we start from the case of curves.

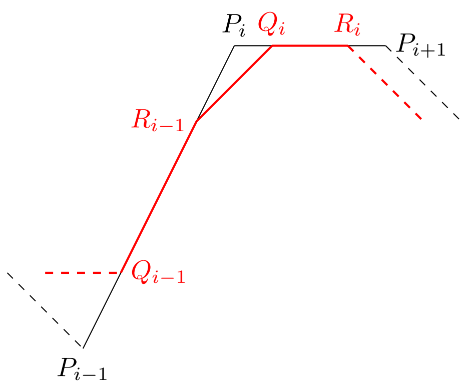
8.1 Charkin's Method (1974)

We start from a piecewise linear curve that connects the points $\dots, P_{i-1}, P_i, P_{i+1}, \dots$. Consider points R_i and Q_i that lie on the segment P_iP_{i+1} such that

$$P_iQ_i = \frac{1}{4} P_iP_{i+1}, \qquad R_iP_{i+1} = \frac{1}{4} P_iP_{i+1}.$$

We produce a new piecewise linear curve that connects the points

$$\dots, Q_{i-1}, R_{i-1}, Q_i, R_i, Q_{i+1}, \dots$$



and iterating this method, it converges to a certain limit, that can be proved to be a quadratic B-spline.

In fact, consider the quadratic uniform B-spline defined by P_i and we insert the middle knot $\widetilde{u} \in [u_k, u_{k-1})$. We have to insert the new points

$$Z_i = (1-\alpha_i)P_{i-1} + \alpha_i P_i, \qquad k-1 \le i \le k, \qquad \alpha_i = \frac{\widetilde{u} - u_i}{u_{i+2} - u_i}$$

$$u_{k+1} - u_k = 2h \implies \widetilde{u} = u_k + h, \qquad \alpha_i = \frac{u_k - u_i + h}{4h}$$

$$\begin{cases} \alpha_k = \frac{1}{4}, & Z_k = \frac{3}{4}P_{k-1} + \frac{1}{4}P_k, \\ \alpha_{k-1} = \frac{3}{4}, & Z_{k-1} = \frac{1}{4}P_{k-2} + \frac{3}{4}P_{k-1}. \end{cases}$$

Let us define the uniform B-spline by convolution:

$$B_0(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the 0-th basis function, and we can translate it by an integer $i \in \mathbb{Z}$. They generate the piecewise constant functions with discontinuous points on the integers. If we use the convolution, we may define the 1-st order B-splines

$$B_1(t) = B_0(t) \star B_0(t) = \begin{cases} t & t \in [0,1), \\ 2-t & t \in [1,2), \\ 0 & \text{otherwise.} \end{cases}$$

They are the hat functions and they generate the piecewise linear functions with discontinuity points on the integers. In general, the p-th order B-splines are defined as

$$B_p(t) = B_{p-1}(t) \star B_0(t) = \int_{\mathbb{R}} B_{p-1}(s) B_0(t-s) ds$$

and they are C^{p-1} functions that are piecewise polynomials of degree at most p and with discontinuities of the p-th derivative on integer points. We can also see them as

$$B_p(t) = B_0(t) \star \cdots \star B_0(t) = \star^p B_0(t)$$

and denote the translations of this function as $B_p(t)^{(i)} := B_p(t-i)$. We notice that

$$B_0(t) = B_0(2t) + B_0(2t-1),$$

and in general we can obtain a *refinement equation.*

Lemma 6.

$$B_p(t) = \frac{1}{2^p} \sum_{k=0}^{p+1} \binom{p+1}{k} B_p(2t-k).$$

Proof.

$$B_p(t) = \star^p B_0(t) = \star^p (B_0(2t) + B_0(2t-1))$$

but the convolution is bilinear, symmetric and

$$f(2t) \star g(2t) = \frac{1}{2}(f \star g)(t), \qquad f(t-i) \star g(t-j) = (f \star g)(t-i-j)$$

so

$$\star^p (B_0(2t) + B_0(2t-1)) = \sum_{k=0}^{p+1} \binom{p+1}{k} (\star^{k-1} B_0(2t-1)) \star (\star^{p-k} B_0(2t))$$
$$= (?)$$

□

Definition 12. A **B-spline curve** of degree p is

$$\mathscr{C}(t) = \sum_{i \in \mathbb{Z}} B_p^{(i)}(t) P_i.$$

8.2 Subdivision Matrix

Let $\mathfrak{B}_p(t)$ be the infinite vector containing all the B-Splines $B_p^{(i)}(t)$.

Definition 13 (Subdivision Matrix)**.** The **Subdivision Matrix** is the infinite matrix S such that

$$\mathfrak{B}_p(t)^T = \mathfrak{B}_p(2t)^T S$$

according to Lemma 6.

We know that

$$s_k := S_{2i+k,i} = \frac{1}{2^p} \binom{p+1}{k}$$

so all columns are the same, but shifted by 2 every time. Notice that the curve can be written as

$$\mathscr{C}(t) = \mathfrak{B}_p(t)^T P,$$

hence

$$\mathscr{C}(t) = \mathfrak{B}_p(t)^T P = \mathfrak{B}_p(2t)^T S P = \cdots = \mathfrak{B}_p(2^j t)^T S^j P.$$

We can thus write the curve as combination of more refined B-splines $\mathfrak{B}_p(2^j t)$ with respect to control points $P^{(j)} := S^j P$, that are

$$P_{2i}^{(j+1)} = \sum_k S_{2i,k} P_k^{(j)} = \sum_{k=i-\lfloor \frac{p+1}{2} \rfloor}^{i} s_{2(i-k)} P_k^{(j)},$$

$$P_{2i+1}^{(j+1)} = \sum_k S_{2i+1,k} P_k^{(j)} = \sum_{k=i-\lfloor \frac{p-1}{2} \rfloor}^{i} s_{2(i-k)+1} P_k^{(j)}.$$

In particular, if $p = 2$,

$$P_{2i}^{(1)} = s_0 P_i + s_2 P_{i-1} = \frac{1}{2}(P_i + P_{i-1}), \qquad P_{2i+1}^{(1)} = s_1 P_i = P_i.$$

In this case, we obtain other points on the segment $P_i P_{i-1}$ and the resulting curve will be piecewise linear. If $p = 4$, we have

$$P_{2i}^{(1)} = s_0 P_i + s_2 P_{i-1} + s_4 P_{i-2} = \frac{1}{8} P_i + \frac{6}{8} P_{i-1} + \frac{1}{8} P_{i-2}, \qquad P_{2i+1}^{(1)} = s_1 P_i + s_3 P_{i-1} = \frac{1}{2}(P_i + P_{i-1}).$$

In this case, it is an approximation scheme that converges to a cubic B-spline. If $p = 3$, we obtain again Charkin's method.

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Let us generalize the subdivision schemes. Suppose we have

- starting control points $P^0 := [P_i^0]_{i \in \mathbb{Z}}$,
- a subdivision scheme $P^{j+1} = S P^j$, where S is row-stochastic and each columns has a finite number of non-zero entries, meaning that the k-th column is 0 outside the rows $2k - a, \ldots, 2k + b$ with fixed a, b,
- each subdivision doubles the number of points, meaning that for every P_k^j we define P_{2k}^{j+1} and P_{2k+1}^{j+1}.

Observe that the control polygon is parametrized by

$$P^j(t) := \sum_{i\in\mathbb{Z}} B_1^{(i)}(2^j t)P_i^j = \mathfrak{B}_1(2^j t)^T P^j.$$

From now on, we use the infinity norm on functions, vectors and matrices. Define the difference operator

$$\Delta = \begin{pmatrix} \ddots & \ddots & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & \ddots \\ & & & & \ddots \end{pmatrix}$$

so that

$$[\Delta P^j]_i = P_{i+1}^j - P_i^j.$$

Lemma 7. *If there exits $c > 0$ and $0 < \gamma < 1$ such that*

$$\|\Delta P^j\| \le c\gamma^j$$

then the function sequence $\{P^j(t)\}_{j\in\mathbb{Z}}$ converges to a continuous function $P^\infty(t)$ for $j \to +\infty$.

Proof.

$$\|P^{j+1}(t) - P^j(t)\| = \|B_1(2^{j+1}t)^T SP^j - B_1(2^j t)^T P^j\|$$

Suppose we call S_1 the matrix for B-splines.

$$\|P^{j+1}(t) - P^j(t)\| = \|B_1(2^{j+1}t)^T SP^j - B_1(2^{j+1}t)^T S_1 P^j\| \le \|B_1(2^{j+1}t)^T\|\|(S-S_1)P_j\| = \|(S-S_1)P_j\|$$

where $\|B_1(2^{j+1}t)^T\| = 1$ since they are B-splines. In this case, there exists a matrix D such that $S - S_1 = D\Delta$.

$$D_{ij} = -\sum_{k=i}^{j}(S-S_1)_{ik} \implies (D\Delta)_{ij} = \sum_{r=-\infty}^{\infty} -\sum_{k=i}^{r}(S-S_1)_{ik}\Delta_{rj} = \sum_{k=i}^{j}(S-S_1)_{ik} - \sum_{k=i}^{j-1}(S-S_1)_{ik} = (S-S_1)_{ij}$$

Moreover $(S - S_1)e = 0$, since they are both row-stochastic, so $\|D\| < \infty$ (?). Consequently,

$$\|P^{j+1}(t) - P^j(t)\| \le \|(S-S_1)P_j\| = \|D\Delta P_j\| \le c\gamma^j\|D\|.$$

We obtain that $P^j(t)$ is a Cauchy sequence, and the uniform convergence is complete, so it uniformly converges to a continuous function $P^\infty(t)$. ☐

Notice that

$$\Delta P^{j+1} = \Delta SP^j = \widehat{D}\Delta P^j$$

where $\widehat{D}$ exists and has finite norm since $\Delta Se = 0$. It means that

$$\|\Delta P^{j+1}\| \le \|\widehat{D}^j\|\|\Delta P^0\|$$

so it converges if $\|\widehat{D}^n\| < 1$ for some $n \in \mathbb{N}$.

References

[1] S. Fallat *Bidiagonal Factorizations of Totally Nonnegative Matrices.* The American Mathematical Monthly, (2001)