Appunti Algebre e Gruppi di Lie

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—> First Lesson Slides

11/03/19 (Gaiffi)

1 Roots

Call $\phi^-(\gamma) := -\phi^+(\gamma)$.

Theorem 1. *Let $\gamma \in E$ regular. The set $\Delta\gamma$ of all indecomponible roots of $\phi^+(\gamma)$ is a basis of ϕ, and every basis is obtainable through this method*

Proof. Step 1 - Every root in $\phi^+(\gamma)$ is a nonnegative linear combination of elements of $\Delta(\gamma)$.

Step 2 - If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) < 0$ except for $\alpha = \beta$.

Step 3 - $\Delta(\gamma)$ is linearly independent.

Step 4 - From Step 1, every element of $\phi^+(\gamma)$ is a nonnegative linear combination of elements of $\Delta(\gamma)$. But $\phi = \phi^+ \cup \phi^-$, ϕ generates E, and $\Delta(\gamma)$ are linearly independent, so $\Delta(\gamma)$ is a basis for E.

Step 5 - Given a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, let $\gamma \in E$ such that $(\gamma, \alpha) > 0$. γ exists since, if we take the projection d_i of α_i on the orthogonal to the space generated by the other α_j, then $d_i \neq 0$ and

$$\gamma = \sum_i r_i d_i, \qquad r_i > 0 \qquad \Longrightarrow \qquad (\gamma, \alpha_j) = r_j(d_j, \alpha_j) > 0.$$

In particular, γ is also regular. Note that ϕ^+ (that are the positive roots with respect to Δ) is a subset of $\phi^+(\gamma)$. As a consequence $\phi^- \subseteq \phi^-(\gamma)$, and by cardinality we can conclude that $\phi^\pm = \phi^\pm(\gamma)$. Moreover $\Delta(\gamma) \subseteq \Delta$ since an element outside δ is not indecomponible. By cardinality, $\Delta(\gamma) = \Delta$. □

Definition 1. The connected components of

$$E - \cup_{\alpha\in\phi} Span(\alpha)^\perp$$

are called **Weyl Rooms**.

Exercise 1. *Every regular γ belong exactly to one Weyl chamber said $C(\gamma)$. Moreover, $C(\gamma) = C(\gamma')$ implies $\Delta(\gamma) = \Delta(\gamma')$.*

The exercise shows that there's a bijective correspondence between chambers and basis. The Weyl group maps chambers on chambers and basis on basis

$$\sigma(C(\gamma)) = C(\sigma\gamma), \qquad \sigma(\Delta(\gamma)) = \Delta(\sigma\gamma).$$

A way to see that $\sigma(\Delta(\gamma))$ is a basis, check that

$$\alpha = \sum_i r_i \alpha_i \implies \sigma(\alpha) = \sum_i r_i \sigma(\alpha_i).$$

The most famous Weyl group is S_n. Other Weyl groups acts like S_n but change the sign to an even number of variables. There are even more that we will discuss later.

Lemma 1. *Let Δ be a basis of a roots system ϕ. If $\alpha \in \phi^+ \setminus \Delta$, then there exists $\beta \in \Delta$ such that $\alpha - \beta \in \phi^+$.*

Proof. If $(\alpha, \beta) \leq 0$ for every $\beta \in \Delta$ (called simple root), using Step 3 of Theorem (1), $\Delta \cup \{\alpha\}$ would be linearly independent, that is a contradiction. There is thus a $\beta \in \Delta$ such that $(\alpha, \beta) > 0$, so (for a precedent Lemma) $\alpha - \beta \in \phi$. Write

$$\alpha = \sum_{\gamma \in \Delta} k_\gamma \gamma$$

with $k_\gamma \geq 0$ and $k_\gamma > 0$ for some $\gamma \neq \beta$ (why $\alpha = k\beta$ is an error?) so $\alpha - \beta \in \phi^+$. (?) □

Corollary 1. *Every $\beta \in \phi^+$ can be written as*

$$\beta = \alpha_1 + \cdots + \alpha_k$$

with $\alpha_i \in \Delta$ even not distinct, such that

$$\alpha_1 + \cdots + \alpha_i \in \phi^+ \qquad \forall i \leq k.$$

Lemma 2. *If $\alpha \in \Delta$ basis, then σ_α permutes $\phi^+ - \{\alpha\}$.*

Proof. $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with $k_\gamma \geq 0$ and $k_\gamma > 0$ for some $\gamma \neq \alpha$. As a consequence

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

so k_γ is still the coefficient of γ and $\sigma_\alpha(\beta) \in \phi^+ 0$. Moreover $\sigma_\alpha(\beta) \neq \alpha$ since $\sigma_\alpha(-\alpha) = \alpha$. □

Corollary 2. *Let*

$$d = \frac{1}{2} \sum_{\beta \in \phi^+} \beta.$$

The following relation holds

$$\sigma_\alpha(d) = d - \alpha \qquad \forall \alpha \in \Delta.$$

Lemma 3. *Let $\alpha_1, \dots, \alpha_t \in \Delta$ not necessarily distinct, with $t \geq 2$. If*

$$\sigma_1 \sigma_2 \dots \sigma_{t-1}(\alpha_t) \in \phi^-$$

then there exists s such that $1 \leq s < t$ and

$$\sigma_1 \sigma_2 \dots \sigma_t = \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1}$$

where $\sigma_i = \sigma_{\alpha_i}$.

Proof. Let

$$\beta_i = \sigma_{i+1} \dots \sigma_{t-1}(\alpha_t).$$

By hypothesis, $\beta_0 \in \phi^-$, and by definition $\beta_{t-1} = \alpha_t \in \phi^+$. Let s be the least index such that $\beta_s \in \phi^+$.

$$\sigma_s(\beta_s) = \beta_{s-1} \in \phi^-.$$

By Lemma we have $\beta_s = \alpha_s$, that is

$$\sigma_{s+1} \dots \sigma_{t-1}(\alpha_t) = \alpha_s.$$

Remember that

$$\sigma_{\tau(\alpha)} = \tau \sigma_\alpha \tau^{-1} \qquad \forall\ \alpha \in \phi, \quad \forall\ \tau \in W.$$

We infer that

$$\sigma_s = \tau \sigma_t \tau^{-1}, \qquad \tau = \sigma_{s+1} \dots \sigma_{t-1}$$

$$\implies \sigma_1 \dots \sigma_{s-1} \sigma_s \sigma_{s+1} \dots \sigma_{t-1} = \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1} \sigma_t$$

$$\implies \sigma_1 \sigma_2 \dots \sigma_t = \sigma_1 \dots \sigma_{s-1} \sigma_{s+1} \dots \sigma_{t-1}$$

since σ_t is an involution. □

Consider A_2 the roots system with three positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$.

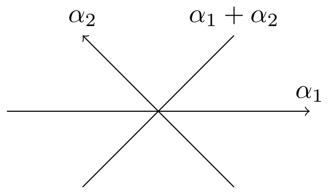

Figure 1: System of roots A_2.

Notice that $s_1 s_2 s_1(\alpha_2) = -\alpha_1$. $\beta_0 = -\alpha_1$, $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_2$, so $s = 1$. By precedent Lemma, we conclude $s_1 s_2 s_1 s_2 = s_2 s_1$.

Corollary 3. *If $\sigma = \sigma_1 \sigma_2 \dots \sigma_t \in W$ is a way to express σ in terms of simple reflections σ_i wrt a basis Δ of least length, then $\sigma(\alpha_t) \in \phi^-$.*

Proof.

$$\sigma(\alpha_t) = \sigma_1 \dots \sigma_{t-1}(-\alpha_t) = -\sigma_1 \dots \sigma_{t-1}(\alpha_t)$$

but if $\sigma(\alpha_t) \in \phi^+$, by Lemma (3), we could write σ as a combination of $t-1$ simple reflections, that is a contradiction. □

2 Weyl Group

Theorem 2. *Let Δ be a basis of ϕ.*

1. *If $\gamma \in E$ is regular, then there exists $\sigma \in W$ such that $(\sigma(\gamma), \alpha) > 0$ for every $\alpha \in \Delta$ (also said, $\sigma(\gamma)$ belong to the Fundamental Chamber). As a consequence W action is transitive on the chambers.*
2. *If Δ' is another basis of ϕ, then there exists $\sigma \in W$ such that $\sigma(\Delta') = \Delta$.*
3. *If $\beta \in \phi$, then there exists $\sigma \in W$ such that $\sigma(\beta) \in \Delta$.*
4. *If $\sigma(\Delta) = \Delta$, then $\sigma = Id$. As a consequence W action is simply transitive on the chambers.*

Proof. Let $W' = (\sigma_\alpha)_{\alpha \in \Delta}$. We will prove 1., 2., 3. for W' and then we will prove $W = W'$.

1. Given $d = \frac{1}{2} \sum_{\beta \in \phi^+} \beta$ and let $\sigma \in W'$ such that $(\sigma(\gamma), d)$ is maximum. If $\alpha \in \Delta$, then $\sigma_\alpha \sigma \in W'$, so

$$(\sigma(\gamma), d) \geq (\sigma_\alpha \sigma(\gamma), d) = (\sigma(\gamma), \sigma_\alpha(d)) = (\sigma(\gamma), d) - (\sigma(\gamma), \alpha)$$

$$\implies (\sigma(\gamma), \alpha) \geq 0 \qquad \forall \alpha \in \Delta.$$

Remember that γ is regular, so

$$(\sigma(\gamma), \alpha) = 0 \implies (\gamma, \sigma^{-1}(\alpha)) = 0$$

that is a contradiction. We conclude that $(\sigma(\gamma), \alpha) > 0$ and thus $\sigma(\gamma)$ belong to the fundamental chamber.

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2. W' acts transitively on basis, since it acts transitively on the chambers, and we know that there's a bijective correspondence between basis and chambers.

3. It is enough to prove that every root belongs to at least one basis. The conclusion then comes from point 2.. Let $\beta \in \phi$ but not in Δ. Note that the hyperplanes $Span(\alpha)^\perp$ are distinct from $Span(\beta)^\perp$ for every $\alpha \neq \pm\beta$. We can choose $\gamma \in Span(\beta)^\perp$ that is not contained in any $Span(\alpha)^\perp$ for $\alpha \neq \pm\beta$. Take

$$m = \min_{\alpha \in \phi \setminus \{\pm\beta\}} |(\gamma, \alpha)| > 0$$

and consider γ' "near" γ (even not in ϕ) such that

$$(\gamma', \beta) = \varepsilon < m, \qquad |(\gamma', \alpha)| > \varepsilon \qquad \forall \alpha \neq \pm\beta.$$

It means that β is indecomponible in $\phi^+(\gamma')$, so $\beta \in \Delta(\gamma')$.

We now prove that $W' = W$. Consider $\beta \in \phi$ and take $\tau \in W'$ such that $\tau(\beta) = \alpha \in \Delta$ from 3..

$$\sigma_\alpha = \sigma_{\tau(\beta)} = \tau \circ \sigma_\beta \circ \tau \in W'.$$

4. Let $\sigma(\Delta) = \Delta$. Let us consider the shortest way to write σ as a composition of simple reflections

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_t.$$

From Corollary 3, $\sigma(\alpha_t) \in \phi^-$, that is a contradiction since $\Delta \subseteq \phi^+$. □

Definition 2. A shortest way to write $\sigma \in W$ as a composition of simple reflections

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_t$$

is called **Reduced Expression** of σ and $l(\sigma) = t$ is its **Length**. In particular $l(Id) = 0$.

Notice that the length of a σ depends on the chosen basis Δ.

Theorem 3. *For every $\sigma \in W$, the length of σ is the number $n(\sigma)$ of positive roots $\beta \in \phi^+$ such that $\sigma(\beta) \in \phi^-$*

Proof. If $l(\sigma) = 0$, then $\sigma = Id$ and $n(\sigma) = 0$.

Suppose by induction that it holds for every τ such that $l(\tau) < l(\sigma)$. Let

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_t$$

be a reduced expression for σ. Take α according to Corollary 3 such that $\sigma(\alpha) \in \phi^-$. Note that σ_α maps $\phi^+ \setminus \{\alpha\}$ in itself and $\alpha \to -\alpha$. Moreover $\sigma(-\alpha) \in \phi^+$ and maps $n(\sigma) - 1$ roots of $\phi^+ \setminus \{\alpha\}$ inside ϕ^-. It means that

$$n(\sigma\sigma_1\alpha) = n(\sigma) - 1, \qquad l(\sigma\sigma_\alpha) = l(\sigma) - 1 \implies n(\sigma) = l(\sigma).$$

□

Exercise 2. *Prove that all the roots $\tau \in \phi^+$ such that $\sigma(\tau) \in \phi^-$ are exactly*

$$\alpha_t, \sigma_t(\alpha_{t-1}), \ldots, \sigma_t \circ \cdots \circ \sigma\sigma_2(\alpha_1).$$

Note that $-\Delta$ is a basis for ϕ. Therefore, there exists an unique $w_0 \in W$ such that $w_0(\Delta) = -\Delta$ and

$$l(w_0) = n(w_0) = |\phi^+|$$

and this is the maximum length possible in W.

For example, taking S_3 referred to the system A_2 (see Figure **??**), we have 3 positive roots and $s_1s_2s_1 = s_2s_1s_2$ are the only element of length 3.

Exercise 3. *Prove that if $w_0(\Delta) = -\Delta$, then w_0 is the only element of maximum length and therefore $w_0 = w_0^{-1}$. Notice that w_0 may not be $-Id$.*

3 Irreducible System

Definition 3. A root system is said to be **Irreducible** if it cannot be partitioned into two proper subsets A, B such that

$$(\alpha, \beta) = 0 \qquad \forall \alpha \in A, \quad \beta \in B$$

For example $A_1 \times A_1$ is reducible (not irreducible).

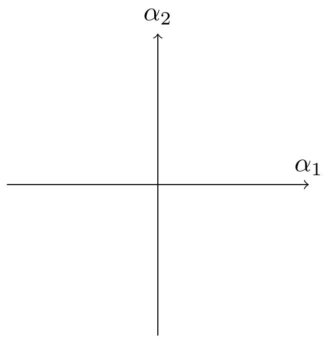

Figure 2: System of roots $A_1 \times A_1$.

Exercise 4. *A root system ϕ is **Irreducible** if and only if any basis Δ cannot be partitioned into two proper subsets A, B such that*

$$(\alpha, \beta) = 0 \qquad \forall \alpha \in A, \quad \beta \in B$$

Lemma 4. *A root system $\phi \subseteq E$ is reducible in an unique way into $\phi_1 \cup \phi_2 \cup \dots \phi_s$ where ϕ_i are irreducible and disjoint. If $E_i = Span(\phi_i)$, then*

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_s$$

where the sum is an orthogonal direct sum.

Lemma 5. *Let ϕ be irreducible. The group W action is irreducible on E and the orbit of every root generates E.*

Proof. Suppose $E' \subseteq E$ is a vector subspace such that $W(E') \subseteq E'$. Given $\alpha \in \phi$,

$$\sigma_\alpha(v) = v - \langle v, \alpha \rangle \alpha$$

so for every $v \in E'$,

$$\alpha \in E' \implies \sigma_\alpha(v) \in E', \qquad \alpha \notin E' \implies \langle v, \alpha \rangle = 0$$

and therefore

$$\phi = (\phi \cap E') \cup (\phi \cap (E')^\perp).$$

We assumed ϕ is irreducible, so $\phi \cap E' = \phi$ and $E' = E$. □

Definition 4. Given a basis $\Delta = \{ \alpha_1, \dots, \alpha_l \}$ of ϕ, let us call

$$(\langle \alpha_i, \alpha_j \rangle)_{i,j}$$

the **Cartan Matrix** of ϕ.

The matrix does not depend on Δ, up to the order of the elements, since all basis are $W-$conjugated. A question that arises here is: given $\phi \subseteq E$ and $\phi' \subseteq E'$ with the same Cartan matrix, are they isomorphic?

Lemma 6. *Yes, and if Δ and Δ' are basis of ϕ, ϕ', with*

$$\Delta = \{ \alpha_1, \dots, \alpha_l \}, \qquad \Delta' = \{ \alpha_1', \dots, \alpha_l' \}$$

and the same associated Cartan matrices, then the bijection $\theta(\alpha_i) = \alpha_i'$ extends to an isomorphism between ϕ and ϕ'.

Proof. If $\alpha, \beta \in \Delta$, we have

$$\sigma_{\theta(\alpha)}(\theta(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle \beta', \alpha' \rangle \alpha' = \theta(\beta) - \langle \beta, \alpha \rangle \theta(\alpha) = \theta(\beta - \langle \beta, \alpha \rangle \alpha) = \theta(\sigma_\alpha(\beta)).$$

W, W' are generated by simple reflections, so $\varphi : W \to W'$ defined as $\Gamma(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ sends Δ in Δ' by $\Gamma(\sigma_{\alpha_i}) = \sigma_{\alpha'_i}$ and it is surjective and invertible, thus an isomorphism. Remember that any $\beta \in \phi$ is conjugated to a simple root that is $\beta = \tau(\alpha_1)$ so

$$\theta(\beta) = (\theta \circ \tau \circ \theta^{-1})(\theta(\alpha)) \in \phi'.$$

As a conclusion, $\theta : \phi \to \phi'$ and it is a bijection. The relation

$$\langle \gamma, \delta \rangle = \langle \theta(\gamma), \theta(\delta) \rangle$$

follows easily. □

4 Coxeter Graphs and Dynkin Diagrams

Given ϕ a root system, a basis Δ and $\alpha, \beta \in \phi^+$ we know that

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3.$$

Definition 5. Let G be a graph with nodes $\alpha_i \in \Delta$ and multi-link where between α_i, α_j there are

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$$

links. It is called **Coxeter Graph**.

Definition 6. Given a Coxeter graph, if two roots have different lengths, we add an arrow towards the shortest. This is called **Dynkin Diagram**

Given a Dynkin Diagram, we can reproduce the Cartan matrix associated, using the rank 2 table.

References