Approximation of the Laplace-Beltrami operator by its symbol

Giovanni Barbarino [1] Davide Bianchi [2] Carlo Garoni [3]

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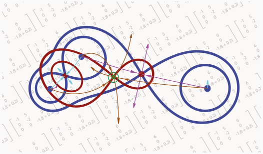

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$1d$ wave equation

Let us consider the following constant-coefficient $1d$ wave equation

$$\begin{cases} \partial_{tt} u - \Delta u = 0, & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u^0(x),\ \partial_t u(0,x) = u^1(x), & x \in (0,1). \end{cases}$$

As a consequence of the Fourier explicit representation formulas, the solutions of this model satisfy the *observability inequality*

$$||(u^0,u^1)||^2_{H^1_0(0,1)\times L^2(0,1)} \leq C_{obs} \int_0^T |\partial_x u(t,1)|^2 dt, \qquad (1)$$

for a suitable constant $C_{obs} = C_{obs}(T) > 0$ provided $T \geq 2$. Inequality (1) ensures that all waves propagating in space-time to reach the extreme $x = 1$ in time $T = 2$.

The spectral gap

Consider the associated $1d$ (spacial) operator

$$
\begin{aligned}
&-\Delta : \mathrm{W}_0^{1,2}((0,1)) \to \mathrm{L}^2((0,1)), \\
&-\Delta[u](x) := -\partial_{xx} u(x).
\end{aligned}
$$

The spectrum $\lambda_k = k^2\pi^2$ satisfies the following gap condition:

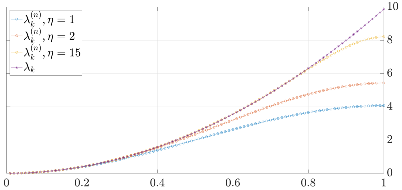
$$
\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \pi = \gamma > 0, \qquad \text{for every } k \geq 1, \tag{2}
$$

The gap condition (2) ensures the boundary observability of the solutions of the associated wave equation in time $T > 2\pi/\gamma$.

Approximation

The classical $2\eta + 1$ points FD approximation $\Delta^{(\eta)}_{\text{dir},n}$ with eigenvalues $\lambda^{(n)}_k$ is such that

$$\min_{k=1,\dots,n-1} \sqrt{\lambda^{(n)}_{k+1}} - \sqrt{\lambda^{(n)}_k} \to 0$$



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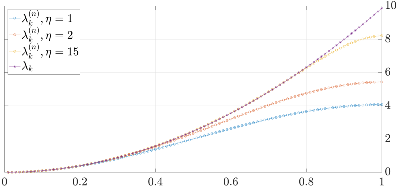
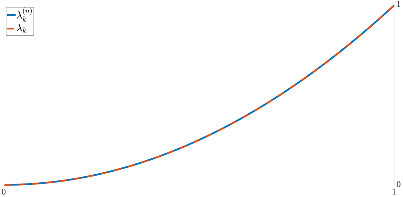


This is fixed equivalently by

- $\Delta^{(\infty)}_{\mathrm{dir},n} = \lim_{\eta\to\infty} \Delta^{(\eta)}_{\mathrm{dir},n}$
- Sinc Collocation
- $T_n(\theta^2) = \left[\widehat{\theta^2}_{i-j}\right]_{i,j=1,\dots,n}$



Variable Coefficient

If $\mathcal{L} = -(a(x)u'(x))'$ on $[0, 1]$ with Dirichlet BC, we have

- No analytic expression of eigenvalues or eigenvectors
- If $a(x) > 0$, then $\lambda_n/n^2 \to \pi^2/B^2,\ B = \int_0^1 a(x)^{-1/2}$

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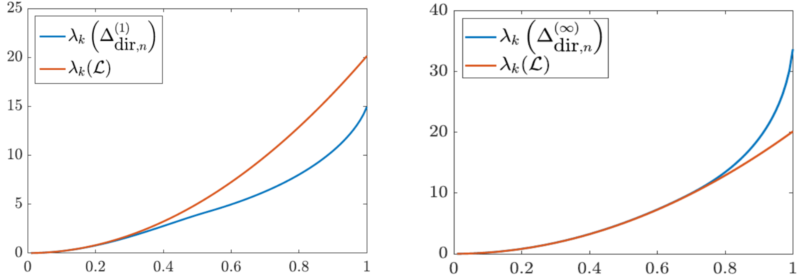
Neither $\Delta^{(\eta)}_{\mathrm{dir},n}$ or its limit has the correct spectral distribution. The same applies also for Iga approximations of any degree. Why?



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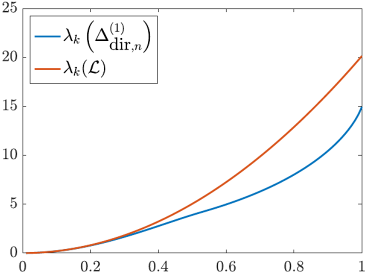

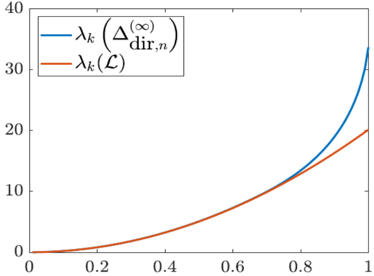

$\Rightarrow$ $\mathcal{L}^{(n)}$ approximation of $\mathcal{L}$ with the right spectral distribution?

Spectral Symbol

Spectral Symbol

The function $\omega : D \subset \mathbb{R}^M \to \mathbb{C}$ is a spectral symbol for $\{A_n\}_n$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\left(\lambda_k\left(A_n\right)\right) = \frac{1}{m(D)} \int_D F\left(\omega(\boldsymbol{y})\right) d\boldsymbol{y}, \quad \forall F \in C_c(\mathbb{C}).$$

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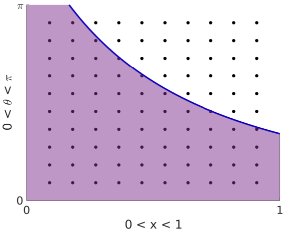

If $\{A_n\}_n \sim_\lambda a(x)\theta^2$ on $[0,1]\times[0,\pi]$, then

$$\Lambda(A_n) \mathrel{\widetilde{=}} \left\{a(ih)(jh\pi)^2 : i,j = 1,\dots,\sqrt{n}\right\}$$

$$\frac{|\{\lambda \in \Lambda(A_n) \cap (-\infty, t)\}|}{n} \to \frac{m\{a(x)\theta^2 < t\}}{\pi}$$

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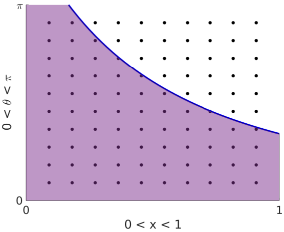

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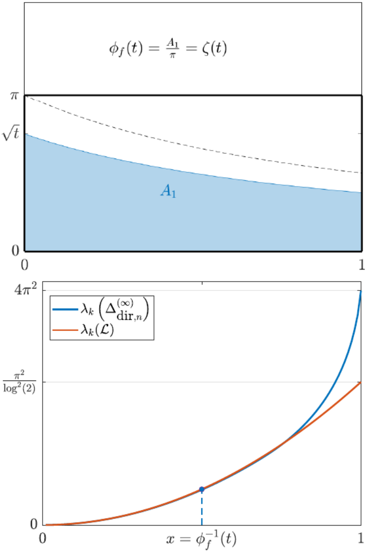
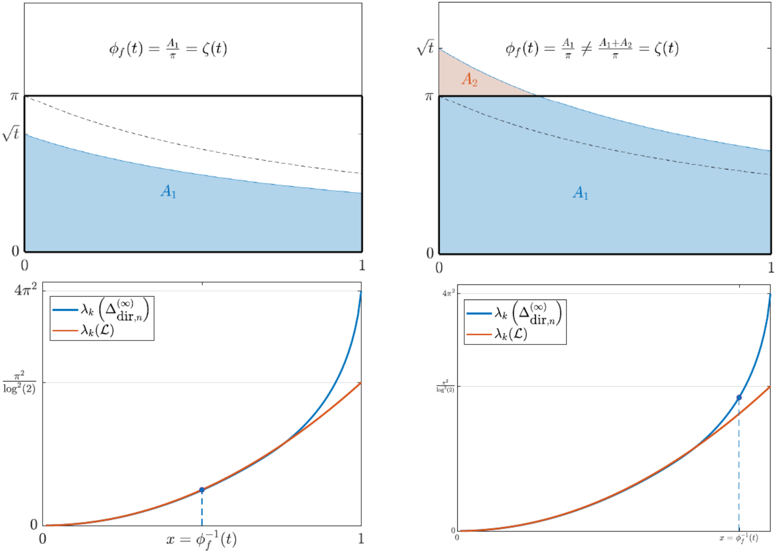
Theorem - See GLT Theory

$\Delta_{\mathrm{dir},n}^{(\infty)}$ and the $\infty-$degree Iga approximations have symbol $a(x)\theta^2$.



$\phi_f(t) = \frac{A_1}{\pi} = \zeta(t)$
π
$\sqrt{t}$
A_1
0
0
1
$4\pi^2$
$\lambda_k\left(\Delta^{(\infty)}_{\mathrm{dir},n}\right)$
$\lambda_k(\mathcal{L})$
$\frac{\pi^2}{\log^2(2)}$
0
0
$x = \phi_f^{-1}(t)$
1



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π
$\sqrt{t}$
A_1
0
0
1
$\phi_f(t) = \frac{A_1}{\pi} \neq \frac{A_1+A_2}{\pi} = \zeta(t)$
$\sqrt{t}$
π
A_2
A_1
0
0
1
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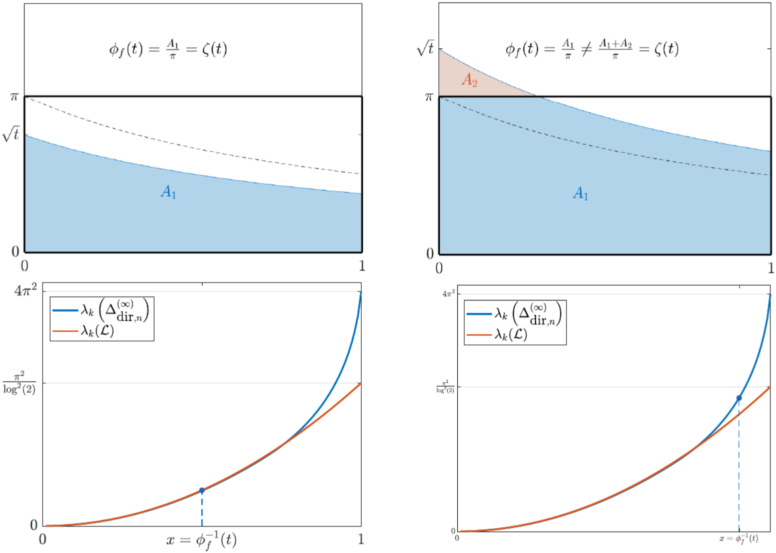

In a sense, $\Lambda(\mathcal{L})$ agrees with $a(x)\theta^2$ on $[0,1] \times [0, \infty)$

Solution 1: Non-Uniform Grids

Theorem - See GLT Theory

Given a diffeomorphism $\phi : [0,1] \to [0,1]$, then a method with symbol $a(x)f(\theta)$ applied to the grid $\{\phi(i/n)\}_{i=1,\dots,n}$ has symbol

$$\frac{a(\phi(x))}{\phi'(x)^{\alpha}} f(\theta)$$

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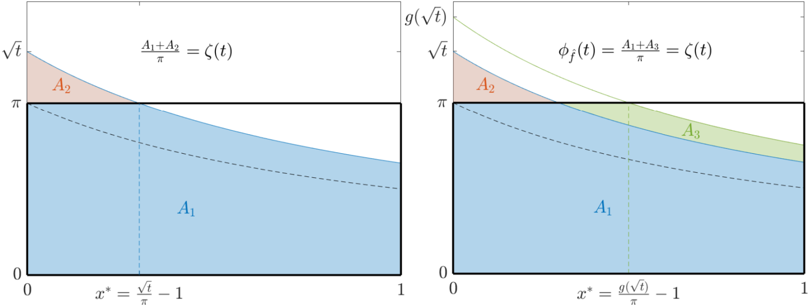
Pros:

- The spectral distribution can be controlled through ϕ
- $\mathcal{L}^{(n)}$ is an approximation of $\mathcal{L}$.

Cons:

- Not always possible to find the right ϕ
  (It has been done for Euler-Cauchy PDEs)

Solution 2: Rearrangement



Find a monotone $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- $\kappa(x, \theta) = \psi(a(x)\theta^2)$ has the right spectral distribution
- $\psi(t) = t$ for any $t \leq \pi^2 \min_x a(x)$

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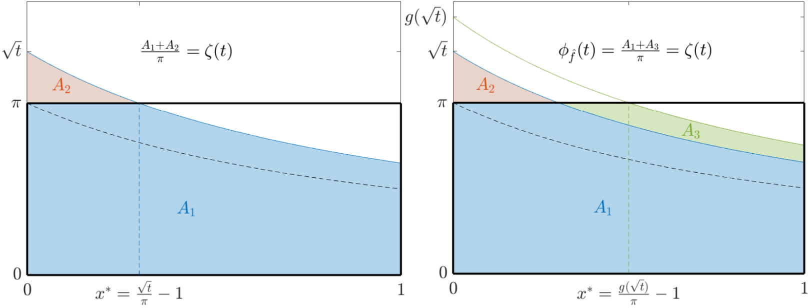

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$$b(w) := m\{x \in [0,1] : a(x) \leq w\} \qquad d(w) := \int_{a(x)\leq w} \sqrt{\frac{1}{a(x)}} dx$$

$$\psi(t) = \frac{1}{B^2}\left[B\sqrt{t} - \sqrt{t}d\left(\frac{t}{\pi^2}\right) + \pi b\left(\frac{t}{\pi^2}\right)\right]^2$$

Teorem - GLT

Given the Fourier series for the real continuous function $\kappa(x, \theta)$

$$\kappa(x, \theta) = \sum_{p=0}^{\infty} c_p(x) \cos(p\theta)$$

then $\kappa(x, \theta)$ is the symbol of the matrix sequence

$$T_n(\kappa(x, \theta)) := \left[\frac{c_{|i-j|}(ih) + c_{|i-j|}(jh)}{2} \right]_{i,j=1,\dots,n}$$

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Open Questions

- T_n is not the only sequence with symbol $\kappa(x,\theta)$.
- Most approximation methods produces $T_n(f(x,\theta))$. For what class of functions, $T_n(f)$ is an approximation of $\mathcal{L}$?

If there's time...

The solutions are valid if the spectral distribution of $\mathcal{L}$ is known

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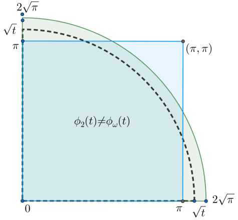
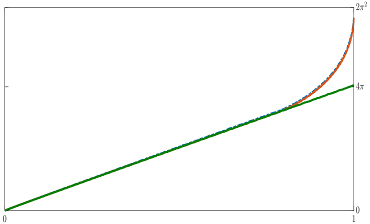
The solutions are valid if the spectral distribution of $\mathcal{L}$ is known

$$\mathcal{L}(u) = -\partial_{xx} u(x,y) - \partial_{yy} u(x,y) \quad \to \quad \lambda_{k_1,k_2} = (k_1^2 + k_2^2)\pi^2$$

Classical approximations give us the symbol $\theta_1^2 + \theta_2^2$ on $[0,\pi]^2$, but again the spectral distributions do not coincide





We can find $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\kappa(\theta_1,\theta_2) = \psi(\theta_1^2 + \theta_2^2)$ that matched the continuous distribution, and

$$T_n(\kappa(\theta_1,\theta_2)) := [c_{i,j}]_{i,j=1,\dots,n} \qquad \kappa(\theta_1,\theta_2) = \sum_{i,j} c_{i,j} \cos(i\theta_1 + j\theta_2)$$

Thank You!

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