Equivalence between GLT sequences and measurable functions

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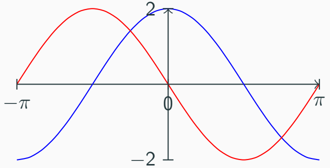
Preliminaries

Spectral Symbol

$$\{A_n\}_n \sim_\sigma k \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{\mu(D)}\int_D F(|k(t)|)\mathrm{dt}$$

$$A_n = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}$$



Approximating Class of Sequence (acs)

$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$ if

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}$$

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for which exist $c(m), \omega(m), n_m$ such that

$$\operatorname{rk} R_{n,m} \le c(m)n \qquad \|N_m\| \le \omega(m) \qquad \forall n > n_m$$

$$\lim_{n\to\infty} c(m) = \lim_{n\to\infty} \omega(m) = 0$$

Properties of GLT Algebra

$\widehat{\mathscr{E}} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n\times n}\}$ $\qquad \mathscr{M}_D = \{k : D \to \mathbb{C}, k \text{ measurable }\}$

$$\widehat{\mathscr{G}} \subseteq \widehat{\mathscr{E}} \times \mathscr{M}_D \quad \text{GLT algebra}$$

Main Properties

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2.
3.
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1. $\widehat{\mathscr{G}}$ is an algebra
2. $\widehat{\mathscr{G}}$ is closed as a pseudometric space into $\widehat{\mathscr{E}} \times \mathscr{M}_D$
3. The GLT symbols are dense into $\mathscr{M}_D$
4. GLT symbols are spectral symbols
   ($\widehat{\mathscr{G}}$ contains $\mathscr{Z}$ the set of zero-distributed sequences)

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More?

Metrics

Completeness of $\mathscr{E}$

The metric d_{acs} on the space $\mathscr{E}$ is defined as

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} p_{acs}(A_n - B_n)$$

$$p_{acs}(A_n - B_n) := \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$$

corresponding to "small rank" R_n and "small norm" N_n

$$A_n - B_n = R_n + N_n$$

Theorem 1

$(\mathscr{E}, d_{acs})$ is a complete metric space.

Idea.

Given a Cauchy sequence $\{B_{n,m}\}_{n,m}$, there exists a map $m(n)$ s.t.

$$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{B_{n,m(n)}\}_n$$

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- The space $\mathscr{M}_D$ is complete with the metric

$$d_m(f,g) = p_m(f-g)$$

$$p_m(f) := \inf_{E\subseteq D} \left\{ \frac{|E^C|}{|D|} + \operatorname*{ess\,sup}_E |f| \right\}$$

Theorem 2

If $\{A_n\}_n \sim_\sigma f$, then

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$f(x)$

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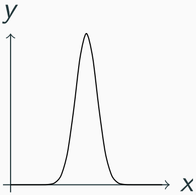

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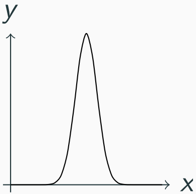

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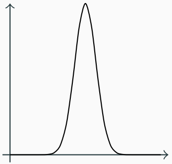

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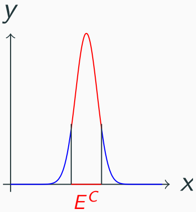

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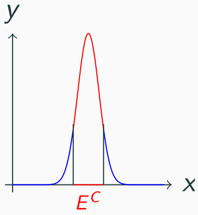

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1. S is a homomorphism of algebras
2. The graph of S into $\mathscr{E} \times \mathscr{M}_D$ is closed
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More?

Let $\{A_n\}_n, \{C_n\}_n \in P_1(\mathscr{G})$.

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$$\implies S(\{A_n\}_n - \{C_n\}_n) = S(\{A_n\}_n) - S(\{C_n\}_n) = k_A - k_C$$

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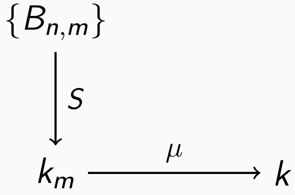

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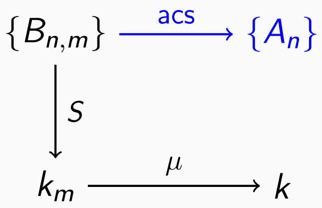

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$\widehat{\mathscr{E}} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n\times n}\}$ $\qquad$ $\mathscr{M}_D = \{k : D \to \mathbb{C}, k \text{ measurable } \}$

$$\begin{array}{ccc} \mathscr{E} & & \mathscr{M}_D \\ \cup\!| & & \cup\!| \\ S : P_1(\mathscr{G}) & \longrightarrow & P_2(\mathscr{G}) \end{array} \qquad \begin{array}{c} \mathscr{E} = \widehat{\mathscr{E}}/\mathscr{Z} \\ \mathscr{G} = \widehat{\mathscr{G}}/\mathscr{Z} \end{array}$$

Main Properties

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More?

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The domain of S is maximal in $\mathscr{E}$

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$\{A_n\}$

- given $\{A_n\}_n$
- find $\{B_{n,m}\}_{n,m}$ GLT sequences with symbols k_m
- if k_m converges, then also $\{B_{n,m}\}_{n,m}$ converges
- if $\{B_{n,m}\}_{n,m}$ converges to $\{A_n\}_n$
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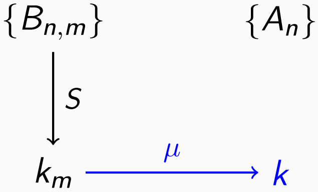

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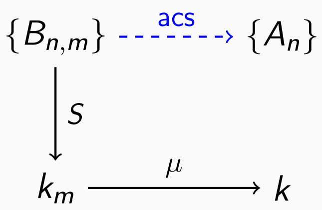

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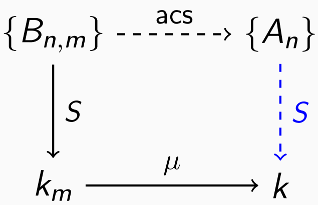

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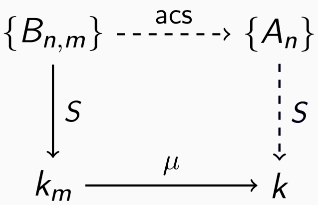

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Metrics on $\mathscr{M}_D$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be an increasing bounded concave and continuous function with $\varphi(0) = 0$

We can define corresponding metrics on $\mathscr{E}$ and $\mathscr{M}_D$

$$p_m^\varphi(f) := \frac{1}{|D|}\int_D \varphi(|f|) \qquad p^\varphi(\{A_n\}_n) := \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \varphi(\sigma_i(A_n))$$

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