# Generalized Locally Toeplitz Sequences: a Link between Measurable Functions and Spectral Symbols 

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## Spectral Symbols

Our Aim

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\left\{\begin{array}{l}
\mathscr{L} u=f \\
B C
\end{array}\right.
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$\xrightarrow{\lg A, \text { Multigrid }}$ FE, FD

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A_{n} u_{n}=f_{n}
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## Our Aim

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\left\{\begin{array}{l}
\mathscr{L} u=f \\
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\end{array} \quad \xrightarrow[\text { FE, FD }]{\text { lgA, Multigrid }} \quad A_{n} u_{n}=f_{n}\right.
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\left\{\begin{array}{l}
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IgA, Multigrid
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$$
\xrightarrow[\text { Quasi-Newton, CG }]{\text { Preconditioned Krylov }}
$$

$$
u_{n}
$$

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\begin{array}{ccc}
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\mathscr{L} u=f \\
B C
\end{array} & \xrightarrow[\text { FE, FD }]{ }
\end{array} \quad A_{n} u_{n}=f_{n}
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## Our Aim

$$
\begin{array}{cc}
\mathscr{L} u=f & \xrightarrow[\text { FE, FD }]{ } \\
B C & \\
A_{n} u_{n}=f_{n} & \begin{array}{l}
\text { Preconditioned Krylov } \\
\text { Quasi-Newton, CG } \\
\uparrow
\end{array} \\
\Lambda\left(A_{n}\right) & A_{n} u_{n}=f_{n}
\end{array}
$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

## Simple Example

$$
\begin{cases}u^{\prime \prime}(x)=f(x) \quad x \in[0,1] \quad \\ u(0)=u(1)=0 & \xrightarrow{F D} \quad A_{n} u_{n}=f_{n}\end{cases}
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## Simple Example

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\begin{aligned}
& \left\{\begin{array}{l}
u^{\prime \prime}(x)=f(x) \\
u(0)=u(1)=0
\end{array} \quad x \in[0,1] \quad \xrightarrow{F D} \quad A_{n} u_{n}=f_{n}\right. \\
& A_{n}=\left[\begin{array}{cccc}
2 & -1 & & \\
-1 & \ddots & \ddots & \\
& \ddots & \ddots & -1 \\
& & -1 & 2
\end{array}\right] \\
& \lambda_{h}\left(A_{n}\right)=2-2 \cos \left(\frac{h \pi}{n+1}\right)
\end{aligned}
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& k(t)=2-2 \cos (t)
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\end{array}\right] \\
& \begin{array}{c}
\lambda_{h}\left(A_{n}\right)= \\
2-2 \cos \left(\frac{2 h \pi}{n+1}-\left\lfloor\left.\frac{2 h}{n+1} \right\rvert\, \frac{\pi}{n+1}\right)\right.
\end{array} \\
& 2-2 \cos (2 t)
\end{aligned}
$$

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& 2-2 \cos \left(\frac{2 h \pi}{n+1}-\left\lfloor\frac{2 h}{n+1}\right\rfloor \frac{\pi}{n+1}\right)
\end{aligned}
$$

$\rightarrow$ The sequence $\left\{A_{n}\right\}_{n}$ has Spectral Symbol $k(t)$

## Asymptotic Distribution

## Spectral Symbol

Let $\left\{A_{n}\right\}_{n}$ a matrix sequence, and $k: D \subseteq \mathbb{R}^{m} \rightarrow \mathbb{C}$ measurable.

$$
\begin{aligned}
& \left\{A_{n}\right\}_{n} \sim_{\lambda} k \Longleftrightarrow \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} F\left(\lambda_{i}\left(A_{n}\right)\right)=\frac{1}{\mu(D)} \int_{D} F(k(t)) \mathrm{dt} \\
& \left\{A_{n}\right\}_{n} \sim_{\sigma} k \Longleftrightarrow \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} F\left(\sigma_{i}\left(A_{n}\right)\right)=\frac{1}{\mu(D)} \int_{D} F(|k(t)|) \mathrm{dt}
\end{aligned}
$$

for all $F \in C_{c}(\mathbb{C})$.

## Asymptotic Distribution

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$$
\frac{\#\left\{i: a<\lambda_{i}\left(A_{n}\right)<b\right\}}{n} \quad \xrightarrow{n \rightarrow \infty} \quad \frac{\mu\{t: a<k(t)<b\}}{\mu(D)}
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Every sequence may have infinite Spectral Symbols

Three Classes of Matrices

Examples of Symbol

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- $Z_{n} \sim_{\sigma} 0$


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## Examples of Symbol

- $Z_{n} \sim_{\sigma} 0$
- $\left\{D_{n}(a)\right\}_{n} \sim_{\lambda, \sigma} a(x)$ where $x \in[0,1]$

$$
\begin{gathered}
a \in C[0,1] \\
D_{n}(a):=\left(\begin{array}{llll}
a(1 / n) & & & \\
& a(2 / n) & & \\
& & a(3 / n) & \\
\\
& & & \ddots
\end{array}\right. \\
\\
\end{gathered}
$$

## Three Classes of Matrices

## Examples of Symbol

- $Z_{n} \sim_{\sigma} 0$
- $\left\{D_{n}(a)\right\}_{n} \sim_{\lambda, \sigma} a(x)$ where $x \in[0,1]$
- $\left\{T_{n}(f)\right\}_{n} \sim_{\sigma} f(\theta)$ where $\theta \in[-\pi, \pi]$

$$
\begin{gathered}
f \in L^{1}[-\pi, \pi] \rightarrow \widehat{f}_{n}=\int_{-\pi}^{\pi} f(\theta) e^{-\mathrm{in} \theta} d \theta \\
T_{n}(f):=\left(\begin{array}{ccccc}
\widehat{f}_{0} & \widehat{f}_{1} & \widehat{f}_{2} & \ldots & \widehat{f}_{n-1} \\
\widehat{f}_{-1} & \widehat{f}_{0} & \ddots & \ddots & \vdots \\
\widehat{f}_{-2} & \ddots & \ddots & \ddots & \widehat{f}_{2} \\
\vdots & \ddots & \ddots & \widehat{f}_{0} & \widehat{f}_{1} \\
\widehat{f}_{-n+1} & \ldots & \widehat{f}_{-2} & \widehat{f}_{-1} & \widehat{f}_{0}
\end{array}\right)
\end{gathered}
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They appear frequently in PDEs

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\begin{aligned}
&\left\{\begin{array}{ll}
\left(a(x) u^{\prime}(x)\right)^{\prime} & =f(x) \quad x \in[0,1] \\
u(0)=u(1)= & 0
\end{array} \quad \xrightarrow{F D} \quad A_{n} u_{n}=f_{n}\right. \\
& A_{n}=D_{n}(a) T_{n}(2-2 \cos (\theta))+Z_{n}
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- The sequence $\left\{A_{n}\right\}_{n}$ has a spectral symbol?
- How do we compute it?


## Space of Matrix Sequences

## a.c.s. Convergence

$$
\widehat{\mathscr{E}}:=\left\{\left\{A_{n}\right\}_{n} \mid A_{n} \in \mathbb{C}^{n \times n}\right\}
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Approximating Class of Sequence [Serra-Capizzano, LAA01]
$\left\{\left\{B_{n, m}\right\}_{n}\right\}_{m} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}_{n}$ if

$$
A_{n}-B_{n, m}=R_{n, m}+N_{n, m}
$$

for which exist $c(m), \omega(m), n_{m}$ such that

$$
\begin{gathered}
\frac{\text { rk } R_{n, m}}{n} \leq c(m) \quad\left\|N_{n, m}\right\| \leq \omega(m) \quad \forall n>n_{m} \\
\lim _{m \rightarrow \infty} c(m)=\lim _{m \rightarrow \infty} \omega(m)=0
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\end{gathered}
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$\rightarrow$ The difference is a sum of small rank and small norm matrices.

$$
\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n} \in \widehat{\mathscr{E}}
$$

## Metric Spaces

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\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n} \in \widehat{\mathscr{E}}
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The a.c.s. convergence is metrizable

$$
\begin{gathered}
d_{a c s}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right)= \\
\lim \sup _{n \rightarrow \infty} \min _{i}\left\{\frac{i-1}{n}+\sigma_{i}\left(A_{n}-B_{n}\right)\right\}
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$$
\begin{gathered}
i \leq j \Longrightarrow \sigma_{i} \geq \sigma_{j} \\
\left\{\sigma_{1}, \sigma_{2}, \ldots, \sigma_{k}, \sigma_{k+1}, \ldots, \sigma_{n-1}, \sigma_{n}\right\}
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## Metric Spaces

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\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n} \in \widehat{\mathscr{E}} \quad f(x), g(x) \in \mathscr{M}_{D}
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$\lim \sup _{n \rightarrow \infty} \min _{i}\left\{\frac{i-1}{n}+\sigma_{i}\left(A_{n}-B_{n}\right)\right\}$

$$
d_{a c s}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right)=
$$

$$
f(x), g(x) \in \mathscr{M}_{D}
$$

The convergence in measure is metrizable

$$
d_{m}(f, g)=
$$

$$
\limsup _{n \rightarrow \infty} \min _{i}\left\{\frac{i-1}{n}+\sigma_{i}\left(A_{n}-B_{n}\right)\right\}
$$

$$
\inf _{z \in \mathbb{R}^{+}}\left\{\frac{\mu\{x:|f(x)-g(x)|>z\}}{\mu(D)}+z\right\}
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Theorem [Barbarino, LAA17]

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Theorem [Barbarino, LAA17]

- $d_{a c s}, d_{m}$ are complete pseudometrics


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Theorem [Barbarino, LAA17]

- $d_{a c s}, d_{m}$ are complete pseudometrics
- $\left\{A_{n}\right\}_{n} \sim_{\sigma} f \Longrightarrow d_{a c s}\left(\left\{A_{n}\right\}_{n},\left\{0_{n}\right\}_{n}\right)=d_{m}(f, 0)$

Closure Property

## Measurable Functions

## Closure Property

Let $\left\{B_{n, m}\right\}_{n} \sim{ }_{\sigma} k_{m}(x)$. Given

1. $k_{m}(x) \xrightarrow{\mu} k(x)$
2. $\left\{A_{n}\right\}_{n} \sim_{\sigma} k(x)$
3. $\left\{B_{n, m}\right\}_{n} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}_{n}$
$\left\{B_{n, m}\right\} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}$
$\downarrow \sim_{\sigma}$
$k_{m} \xrightarrow{\mu} k$

## Measurable Functions

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we have (1), (3) $\Longrightarrow$ (2).


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3. $\left\{B_{n, m}\right\}_{n} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}_{n}$
we have (1), (3) $\Longrightarrow$ (2).

## Problems

## Measurable Functions

## Closure Property

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The Spectral Symbol is Not Unique

GLT Sequences

GLT Space

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\widehat{\mathscr{G}} \subseteq \widehat{\mathscr{E}} \times \mathscr{M}_{D}
$$

where $D=[0,1] \times[-\pi, \pi]$

## GLT Space

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- $\left\{T_{n}(f)\right\}_{n} \sim_{\sigma} f(\theta) \quad f(\theta) \in L^{1}[-\pi, \pi]$
- $\left\{D_{n}(a)\right\}_{n} \sim_{\lambda, \sigma} a(x) \quad a(x) \in C([0,1])$
- $Z_{n} \sim_{\sigma} 0$


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\widehat{\mathscr{G}} \subseteq \widehat{\mathscr{E}} \times \mathscr{M}_{D}
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The algebra generated by $L^{1}[-\pi, \pi]$ and $C([0,1])$ is dense in $\mathscr{M}_{D}$.

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GLT Algebra [Serra-Capizzano, LAA03]
The GLT Space is the smallest closed algebra with respect to $d_{a c s} \times d_{m}$ that contains

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$$
\left\{A_{n}+B_{n}\right\}_{n} \sim_{G L T} k_{1}+k_{2} \quad\left\{A_{n} B_{n}\right\}_{n} \sim_{G L T} k_{1} k_{2}
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The GLT symbol is always Unique and a Spectral Symbol

## Three Classes of Matrices

## Examples of Symbol

- $Z_{n} \sim_{G L T} 0$
- $\left\{D_{n}(a)\right\}_{n} \sim_{G L T} a(x)$ where $a(x) \in C([0,1])$
- $\left\{T_{n}(f)\right\}_{n} \sim_{G L T} f(\theta)$ where $f(\theta) \in L^{1}[-\pi, \pi]$

They appear frequently in PDEs

$$
\begin{aligned}
& \begin{cases}\left(a(x) u^{\prime}(x)\right)^{\prime} & =f(x) \quad x \in[0,1] \quad \xrightarrow{F D} \quad A_{n} u_{n}=f_{n} \\
u(0)=u(1)= & 0\end{cases} \\
& A_{n}=D_{n}(a) T_{n}(2-2 \cos (\theta))+Z_{n}
\end{aligned}
$$

- The sequence $\left\{A_{n}\right\}_{n}$ has a spectral symbol?
- How do we compute it?


## Three Classes of Matrices

## Examples of Symbol

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&\left\{A_{n}\right\}_{n} \sim_{G L T} a(x)(2-2 \cos (\theta))
\end{aligned}
$$

## GLT properties

$\widehat{\mathscr{E}}:=\left\{\left\{A_{n}\right\}_{n}: A_{n} \in \mathbb{C}^{n \times n}\right\} \quad \mathscr{M}_{D}=\{k: D \rightarrow \mathbb{C}, k$ measurable $\}$

$$
\begin{gathered}
\widehat{\mathscr{E}} \\
\cup 1 \\
P_{1}(\widehat{\mathscr{G}})
\end{gathered}
$$

$$
\begin{gathered}
\mathscr{M}_{D} \\
\cup 1 \\
P_{2}(\widehat{\mathscr{G}})
\end{gathered}
$$

## Main Properties

1. $\widehat{\mathscr{G}}$ is an algebra
2. $\widehat{\mathscr{G}}$ is closed as a pseudometric space into $\widehat{\mathscr{E}} \times \mathscr{M}_{D}$
3. GLT symbols are spectral symbols
( $\widehat{\mathscr{G}}$ contains $\mathscr{Z}$ the set of zero-distributed sequences)

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\widehat{\mathscr{E}} & \mathscr{M}_{D} \\
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$\left(\operatorname{ker}(\widehat{S})=P_{1}(\mathscr{Z})\right)$

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\mathscr{E} & \mathscr{M}_{D} & \mathscr{E}=\widehat{\mathscr{E}} / \mathscr{Z} \\
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S: P_{1}(\mathscr{G}) \longrightarrow \mathscr{G}=\widehat{\mathscr{G}} / \mathscr{Z}
\end{array}
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( $S$ is injective)

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More?

## Identification

Let $\left\{A_{n}\right\}_{n},\left\{C_{n}\right\}_{n} \in P_{1}(\mathscr{G})$.

1. $S$ homomorphism of algebras

Let $\left\{A_{n}\right\}_{n},\left\{C_{n}\right\}_{n} \in P_{1}(\mathscr{G})$.

1. $S$ homomorphism of algebras

$$
\Longrightarrow S\left(\left\{A_{n}\right\}_{n}-\left\{C_{n}\right\}_{n}\right)=S\left(\left\{A_{n}\right\}_{n}\right)-S\left(\left\{C_{n}\right\}_{n}\right)=k_{A}-k_{C}
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$$

4. $\left\{A_{n}\right\}_{n} \sim_{\sigma} S\left(\left\{A_{n}\right\}_{n}\right)$

$$
\Longrightarrow\left\{A_{n}\right\}_{n}-\left\{C_{n}\right\}_{n} \sim_{\sigma} k_{A}-k_{C}
$$

Th2. $\left\{A_{n}\right\}_{n} \sim_{\sigma} f \Longrightarrow d_{\text {acs }}\left(\left\{A_{n}\right\}_{n},\left\{0_{n}\right\}_{n}\right)=d_{m}(f, 0)$

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$$
\begin{aligned}
\Longrightarrow d_{a c s}\left(\left\{A_{n}\right\}_{n},\left\{C_{n}\right\}_{n}\right) & =d_{a c s}\left(\left\{A_{n}\right\}_{n}-\left\{C_{n}\right\}_{n},\left\{0_{n}\right\}_{n}\right) \\
=d_{m}\left(k_{A}-k_{C}, 0\right) & =d_{m}\left(k_{A}, k_{C}\right)
\end{aligned}
$$

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$S$ is an isometry

Let $k \in \mathscr{M}_{D}$
Let $k \in \mathscr{M}_{D}$ and $k_{m} \xrightarrow{\mu} k$ such that exist $S\left(\left\{B_{n, m}\right\}\right)=k_{m}$
Iso. $S$ is an isometry
$\longrightarrow d \operatorname{ISR}\}\{E\},)=d_{m}\left(k_{s}, k_{f}\right) \longrightarrow\left\{B_{n, m}\right\}$ Cauchy
Th1. $\mathscr{E}$ is complete $\Longrightarrow \exists\left\{A_{n}\right\}_{n}:\left\{B_{n, m}\right\}_{n, m} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}_{n}$
2. The graph of $S$ into $\mathscr{E} \times \mathscr{M}_{n}$ is closed $\Longrightarrow S\left(\left\{A_{m}\right\}_{n}\right)=k$

$$
k
$$

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$$
\begin{aligned}
& \left\{B_{n, m}\right\} \\
& \qquad \begin{array}{l} 
\\
k_{m} \\
S
\end{array} \quad \mu
\end{aligned}
$$

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$$
\Longrightarrow d_{a c s}\left(\left\{B_{n, s}\right\},\left\{B_{n, r}\right\}\right)=d_{m}\left(k_{s}, k_{r}\right) \Longrightarrow\left\{B_{n, m}\right\} \text { Cauchy }
$$

$$
\underset{k_{m}}{\substack{\left\{B_{n, m}\right\} \\ k_{m} \\ \mu}}
$$

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$$
\begin{gathered}
\left\{B_{n, m}\right\} \xrightarrow{\text { acs }}\left\{A_{n}\right\} \\
\qquad \begin{array}{l}
\text { } \\
k_{m} \xrightarrow{\mu} k
\end{array}
\end{gathered}
$$

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$$
\begin{aligned}
& \left\{B_{n, m}\right\} \xrightarrow{\text { acs }}\left\{A_{n}\right\} \\
& \underset{k_{m}}{\downarrow} \quad \stackrel{\downarrow}{ }{ }^{\downarrow}
\end{aligned}
$$

Let $k \in \mathscr{M}_{D}$
Let $k \in \mathscr{M}_{D}$ and $k_{m} \xrightarrow{\mu} k$ such that exist $S\left(\left\{B_{n, m}\right\}\right)=k_{m}$
Iso. $S$ is an isometry

$$
\Longrightarrow d_{a c s}\left(\left\{B_{n, s}\right\},\left\{B_{n, r}\right\}\right)=d_{m}\left(k_{s}, k_{r}\right) \Longrightarrow\left\{B_{n, m}\right\} \text { Cauchy }
$$

Th1. $\mathscr{E}$ is complete $\Longrightarrow \exists\left\{A_{n}\right\}_{n}:\left\{B_{n, m}\right\}_{n, m} \xrightarrow{\text { a.c.s. }}\left\{A_{n}\right\}_{n}$
2. The graph of $S$ into $\mathscr{E} \times \mathscr{M}_{D}$ is closed $\Longrightarrow S\left(\left\{A_{n}\right\}_{n}\right)=k$

$\widetilde{\operatorname{Im}}(S)$ is closed in $\mathscr{M}_{D}$
$\widehat{\mathscr{E}}:=\left\{\left\{A_{n}\right\}_{n}: A_{n} \in \mathbb{C}^{n \times n}\right\} \quad \mathscr{M}_{D}=\{k: D \rightarrow \mathbb{C}, k$ measurable $\}$

$$
\begin{array}{ccc}
\mathscr{E} & \mathscr{M}_{D} & \mathscr{E}=\widehat{\mathscr{E}} / \mathscr{Z} \\
\cup \| & \cup \| & P_{2}(\mathscr{G})
\end{array}
$$

## Main Properties

1. $S$ is a homomorphism of algebras
2. The graph of $S$ into $\mathscr{E} \times \mathscr{M}_{D}$ is closed
3. $\left\{A_{n}\right\}_{n} \sim_{\sigma} S\left(\left\{A_{n}\right\}_{n}\right)$

More?
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$S$ is an isometry and $\widetilde{I m}(S)$ is closed

We know that, for GLT, $\widetilde{\operatorname{Im}}(S)$ is dense in $\mathscr{M}_{D}$, so
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## Main Properties

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$$
\mathscr{G} \cong \mathscr{M}_{D}
$$

[Barbarino, LAA17]

## $\left\{A_{n}\right\}$

- given $\left\{A_{n}\right\}_{n}$ find $\left\{B_{n, m}\right\}_{n, m}$ GLT sequences with symbols $k_{m}$ if $k_{m}$ converges, then also $\left\{B_{n, m}\right\}_{n, m}$ converges

$$
\begin{array}{cc}
\left\{B_{n, m}\right\} & \left\{A_{n}\right\} \\
\downarrow_{m} & \\
k_{m} &
\end{array}
$$

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$$
\begin{array}{rll}
\left\{B_{n, m}\right\} & & \left\{A_{n}\right\} \\
\qquad \begin{array}{lll}
\downarrow \\
k_{m} & & \\
& & \\
& &
\end{array}
\end{array}
$$

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$$
\begin{aligned}
& \left\{B_{n, m}\right\} \ldots \text { acs }->\left\{A_{n}\right\} \\
& \underset{k_{m}}{\downarrow} \xrightarrow{\downarrow} k
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- if $\left\{B_{n, m}\right\}_{n, m}$ converges to $\left\{A_{n}\right\}_{n}$

$$
\begin{array}{cc}
\left\{B_{n, m}\right\}-\operatorname{acs} & \left\{A_{n}\right\} \\
\qquad & \\
k_{m} \xrightarrow{\mu} & S
\end{array}
$$

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- Then $\left\{A_{n}\right\}_{n}$ has spectral symbol $k$


## $\left\{B_{n, m}\right\} \cdots \underset{\text { acs }}{ }\left\{A_{n}\right\}$ <br> 

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- if $\left\{B_{n, m}\right\}_{n, m}$ converges to $\left\{A_{n}\right\}_{n}$
- Then $\left\{A_{n}\right\}_{n}$ has spectral symbol $k$
$\longrightarrow$ proving the acs convergence is difficult


## Metrics on $\mathscr{M}_{D}$

Let $\varphi: \mathbb{R}^{+} \rightarrow \mathbb{R}$ be an increasing bounded concave and continuous function with $\varphi(0)=0$

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We can define corresponding metrics on $\mathscr{E}$ and $\mathscr{M}_{D}$

$$
\begin{array}{ll}
p_{m}^{\varphi}(f):=\frac{1}{|D|} \int_{D} \varphi(|f|) & p^{\varphi}\left(\left\{A_{n}\right\}_{n}\right):=\limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \varphi\left(\sigma_{i}\left(A_{n}\right)\right) \\
d_{m}^{\varphi}(f, g):=p_{m}^{\varphi}(f-g) & d^{\varphi}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right):=p^{\varphi}\left(\left\{A_{n}-B_{n}\right\}_{n}\right)
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\end{array}
$$

Theorem 3 [Barbarino, Garoni, '17] $d^{\varphi}$ is a complete metric on $\mathscr{E}$ inducing the acs convergence.

$$
\left\{A_{n}\right\}_{n} \sim_{\sigma} f \Longrightarrow p^{\varphi}\left(\left\{A_{n}\right\}_{n}\right)=p_{m}^{\varphi}(f)
$$

$\left\{A_{n}\right\}_{n} \sim_{G L T} k,\left\{B_{n}\right\}_{n} \sim_{G L T} h \Longrightarrow d^{\varphi}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right)=d_{m}^{\varphi}(k, h)$

## Metrics on $\mathscr{M}_{D}$

Concave functions

- $\varphi_{1}(x)=\min \{x, 1\}$
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& d_{1}^{\varphi}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right)=\limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \min \left\{\sigma_{i}\left(A_{n}-B_{n}\right), 1\right\} \\
& d_{2}^{\varphi}\left(\left\{A_{n}\right\}_{n},\left\{B_{n}\right\}_{n}\right)=\limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_{i}\left(A_{n}-B_{n}\right)}{\sigma_{i}\left(A_{n}-B_{n}\right)+1}
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\end{aligned}
$$

$\longrightarrow$ New ways to test the acs convergence

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