

Generalized Locally Toeplitz Sequences: a Link between Measurable Functions and Spectral Symbols

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Spectral Symbols

Our Aim

$$\left\{ \begin{array}{l} \mathcal{L}u = f \\ BC \end{array} \right. \xrightarrow[\text{FE, FD}]{\text{IgA, Multigrid}} A_n u_n = f_n$$

$$A_n u_n = f_n \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} u_n$$

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 $\Lambda(A_n)$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

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Simple Example

$$\begin{cases} u''(x) = f(x) & x \in [0, 1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & \end{bmatrix}$$

$$\lambda_h(A_n) = 2 - 2 \cos\left(\frac{h\pi}{n+1}\right)$$

→ The sequence $\{A_n\}_n$ has Spectral Symbol $k(t)$

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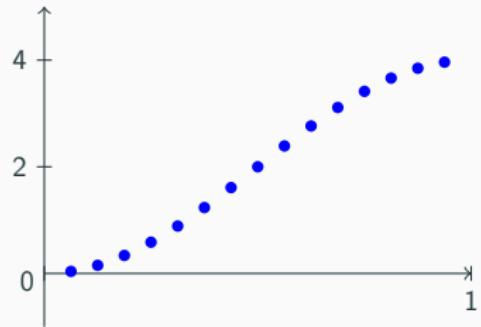
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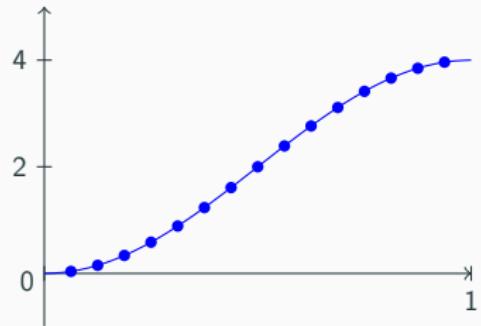
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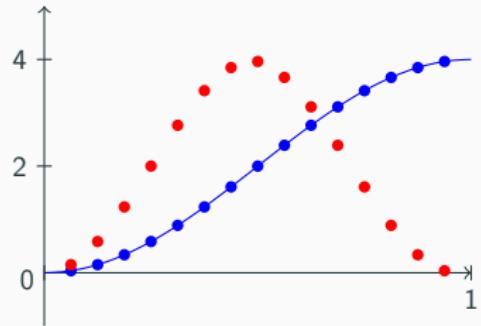
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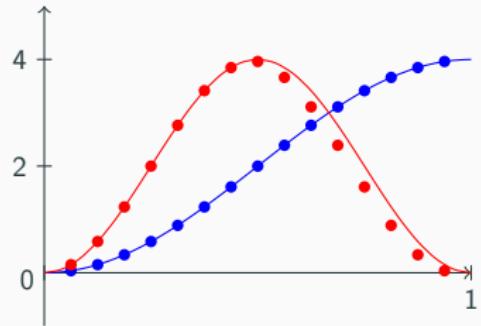
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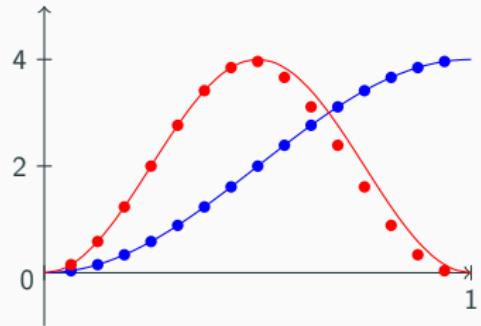
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Asymptotic Distribution

Spectral Symbol

Let $\{A_n\}_n$ a matrix sequence, and $k : D \subseteq \mathbb{R}^m \rightarrow \mathbb{C}$ measurable.

$$\{A_n\}_n \sim_{\lambda} k \iff \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t)) dt$$

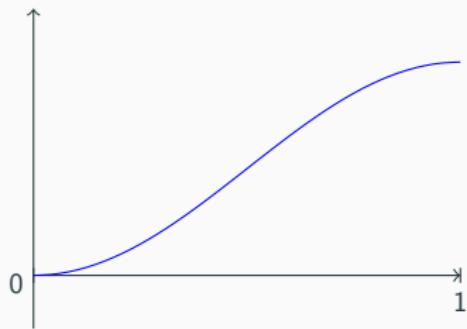
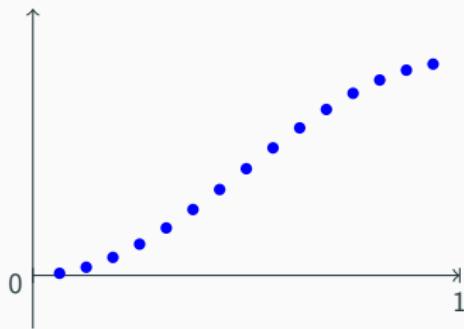
$$\{A_n\}_n \sim_{\sigma} k \iff \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n F(\sigma_i(A_n)) = \frac{1}{\mu(D)} \int_D F(|k(t)|) dt$$

for all $F \in C_c(\mathbb{C})$.

Every sequence may have infinite Spectral Symbols

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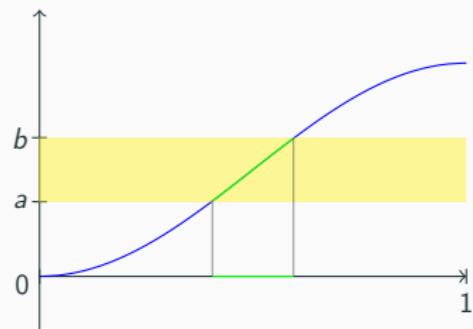
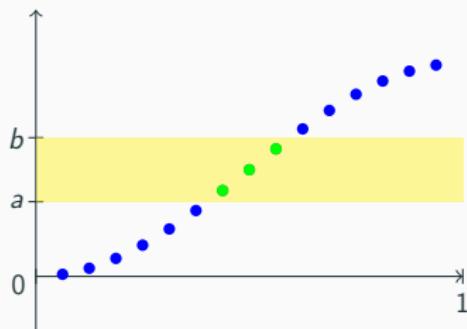


$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n \rightarrow \infty} \frac{\mu\{t : a < k(t) < b\}}{\mu(D)}$$

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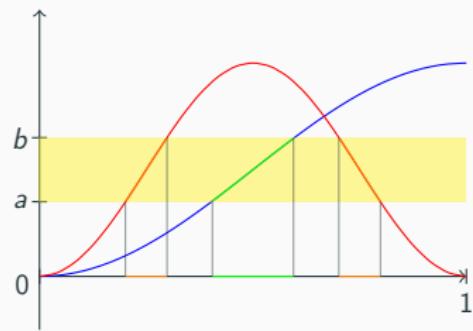
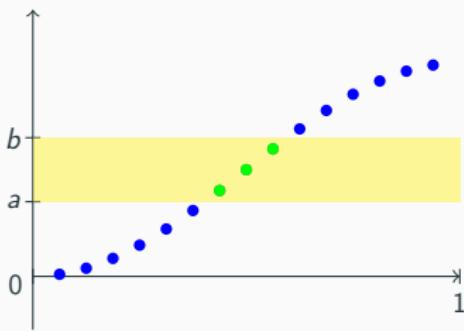


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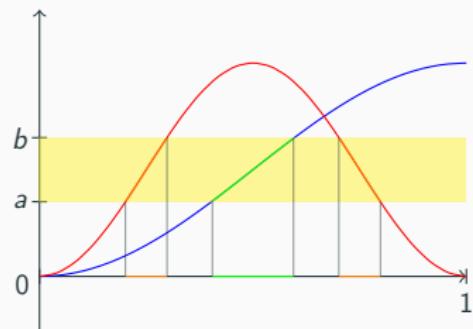
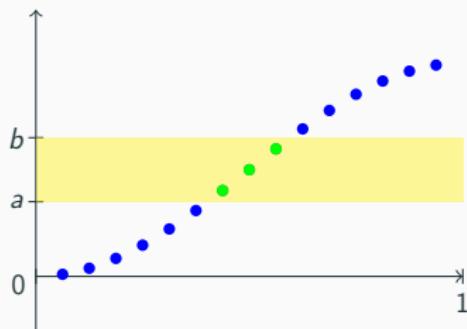


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Three Classes of Matrices

Examples of Symbol

- $Z_n \sim_{\sigma} 0$
- $\{D_n(a)\}_n \sim_{\lambda, \sigma} a(x)$ where $x \in [0, 1]$
- $\{T_n(f)\}_n \sim_{\sigma} f(\theta)$ where $\theta \in [-\pi, \pi]$

$$a \in C[0, 1]$$

$$D_n(a) := \begin{pmatrix} a(1/n) & & & & \\ & a(2/n) & & & \\ & & a(3/n) & & \\ & & & \ddots & \\ & & & & a(1) \end{pmatrix}$$

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$$f \in L^1[-\pi, \pi] \rightarrow \hat{f}_n = \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

$$T_n(f) := \begin{pmatrix} \hat{f}_0 & \hat{f}_1 & \hat{f}_2 & \dots & \hat{f}_{n-1} \\ \hat{f}_{-1} & \hat{f}_0 & \ddots & \ddots & \vdots \\ \hat{f}_{-2} & \ddots & \ddots & \ddots & \hat{f}_2 \\ \vdots & \ddots & \ddots & \hat{f}_0 & \hat{f}_1 \\ \hat{f}_{-n+1} & \dots & \hat{f}_{-2} & \hat{f}_{-1} & \hat{f}_0 \end{pmatrix}$$

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They appear frequently in PDEs

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0, 1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = D_n(a) T_n(2 - 2 \cos(\theta)) + Z_n$$

- The sequence $\{A_n\}_n$ has a spectral symbol?
- How do we compute it?

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Space of Matrix Sequences

a.c.s. Convergence

$$\widehat{\mathcal{E}} := \{\{A_n\}_n \mid A_n \in \mathbb{C}^{n \times n}\}$$

Approximating Class of Sequence [Serra-Capizzano, LAA01]

$$\{\{B_{n,m}\}_n\}_m \xrightarrow{\text{a.c.s.}} \{A_n\}_n \text{ if}$$

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

for which exist $c(m), \omega(m), n_m$ such that

$$\frac{\operatorname{rk} R_{n,m}}{n} \leq c(m) \quad \|N_{n,m}\| \leq \omega(m) \quad \forall n > n_m$$

$$\lim_{m \rightarrow \infty} c(m) = \lim_{m \rightarrow \infty} \omega(m) = 0$$

→ The difference is a sum of **small rank** and **small norm** matrices.

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Metric Spaces

$$\{A_n\}_n, \{B_n\}_n \in \widehat{\mathcal{E}}$$

$$f(x), g(x) \in \mathcal{M}_D$$

The a.c.s. convergence is
metrizable

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n \rightarrow \infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$$

The convergence in measure is
metrizable

$$d_m(f, g) = \inf_{z \in \mathbb{R}^+} \left\{ \frac{\mu\{x : |f(x) - g(x)| > z\}}{\mu(D)} + z \right\}$$

$$i \leq j \implies \sigma_i \geq \sigma_j$$

$$\{\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{n-1}, \sigma_n\}$$



Theorem [Barbarino, LAA17]

Given $\{A_n\}_n, \{B_n\}_n \in \widehat{\mathcal{E}}$ and $f, g \in \mathcal{M}_D$

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = d_m(f, g) \iff \lim_{n \rightarrow \infty} \mu\{x : |f(x) - g(x)| > \epsilon\} = 0$$

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Theorem [Barbarino, LAA17]

Given $\{A_n\}_n$, $\{B_n\}_n$ measurable sets and σ_i a sequence of numbers

$$\lim_{n \rightarrow \infty} d_{acs}(\{A_n\}_n, \{B_n\}_n) = \lim_{n \rightarrow \infty} d_m(f_n, g_n)$$

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Theorem [Barbarino, LAA17]

Given $\{A_n\}_n$ and $\{B_n\}_n$ in $\widehat{\mathcal{E}}$, if $d_{acs}(A_n, B_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\mu(A_n) \rightarrow \mu(B_n)$.

Given $f, g \in \mathcal{M}_D$, if $d_m(f, g) \rightarrow 0$ as $n \rightarrow \infty$, then $\mu(f^{-1}(A)) \rightarrow \mu(g^{-1}(A))$.

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$$i \leq j \implies \sigma_i \geq \sigma_j$$
$$\{\color{red}\sigma_1, \sigma_2, \dots, \sigma_k\color{black}, \color{blue}\sigma_{k+1}, \dots, \sigma_{n-1}, \sigma_n\color{black}\}$$



Theorem [Barbarino, LAA17]

If $\{A_n\}_n$ is a sequence of sets in a metric space (X, d) such that

$\lim_{n \rightarrow \infty} \sigma_i(A_n) = 0$ for all $i \in \mathbb{N}$, then $\{A_n\}_n$ converges to the empty set in the a.c.s.

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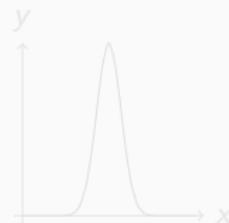
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$$d_m(f, g) = \inf_{z \in \mathbb{R}^+} \left\{ \frac{\mu\{x : |f(x) - g(x)| > z\}}{\mu(D)} + z \right\}$$

$$i \leq j \implies \sigma_i \geq \sigma_j$$
$$\{\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{n-1}, \sigma_n\}$$



Theorem [Barbarino, LAA17]

If $\{A_n\}_n$ is a sequence of sets in a metric space (X, d) and $\{B_n\}_n$ is a sequence of sets in (Y, d') , then

$\{A_n\}_n$ converges in measure to A if and only if $\{f^{-1}(A_n)\}_n$ converges in measure to $f^{-1}(A)$.

Metric Spaces

$$\{A_n\}_n, \{B_n\}_n \in \widehat{\mathcal{E}}$$

The a.c.s. convergence is
metrizable

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n \rightarrow \infty} \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$$

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$$f(x), g(x) \in \mathcal{M}_D$$

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Theorem [Barbarino, LAA17]

Given two sequences of sets $\{A_n\}_n$ and $\{B_n\}_n$ such that $A_n \subseteq B_n$ for all n , then

- $\{A_n\}_n$ converges in measure to f if and only if $\{B_n\}_n$ converges in measure to f .
- $\{A_n\}_n$ converges almost surely to f if and only if $\{B_n\}_n$ converges almost surely to f .

Metric Spaces

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The a.c.s. convergence is
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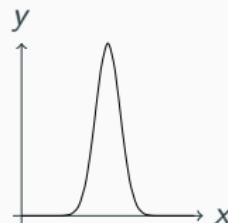
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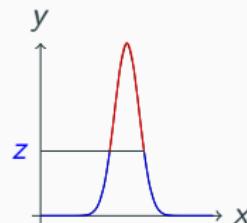
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Theorem [Barbarino, LAA17]

d_{acs} and d_m are complete pseudometrics

Proof: <https://www.mathworksheetsland.com/10/10metric.pdf>

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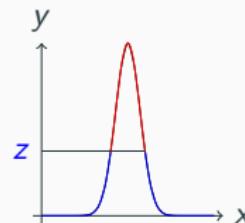
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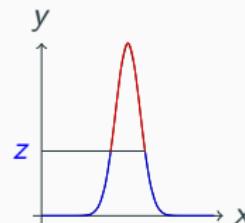
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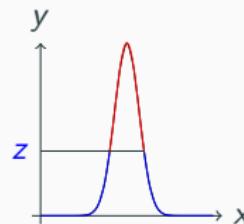
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Measurable Functions

Closure Property

Let $\{B_{n,m}\}_n \sim_\sigma k_m(x)$. Given

$$1. \ k_m(x) \xrightarrow{\mu} k(x)$$

$$2. \ \{A_n\}_n \sim_\sigma k(x)$$

$$3. \ \{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$$

we have (1), (3) \Rightarrow (2).

$$\begin{array}{ccc} \{B_{n,m}\} & \xrightarrow{\text{a.c.s.}} & \{A_n\} \\ \downarrow \sim_\sigma & & \\ k_m & \xrightarrow{\mu} & k \end{array}$$

From the above diagram, we can see that $\{B_{n,m}\}_n \sim_\sigma k_m(x)$ and $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

Therefore, $\{B_{n,m}\}_n \sim_\sigma k_m(x) \Rightarrow \{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

The Spectral Symbol is Not Unique

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Problems

1. If $\{f_n\}_n$ is a sequence of measurable functions on a set E , show that the limit function f is measurable.

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- $\{A_n\}_n \sim_\sigma f, \{B_n\}_n \sim_\sigma g \not\Rightarrow d_{acs}(\{A_n\}_n, \{B_n\}_n) = d_m(f, g)$
- Stronger, (1), (2) $\not\Rightarrow$ (3) (3), (2) $\not\Rightarrow$ (1)

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GLT Sequences

GLT Space

$$\widehat{\mathcal{G}} \subseteq \widehat{\mathcal{E}} \times \mathcal{M}_D$$

where $D = [0, 1] \times [-\pi, \pi]$

- $\{T_n(f)\}_n \sim f(\theta) \quad f(\theta) \in L^1[-\pi, \pi]$
- $\{D_n(a)\}_n \sim a(x) \quad a(x) \in C([0, 1])$
- $Z_n \sim 0$

The algebra generated by $L^1[-\pi, \pi]$ and $C([0, 1])$ is dense in \mathcal{M}_D .

Therefore, \mathcal{G} is a GLT space.

It is also a Banach space, since it is closed under addition and scalar multiplication.

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Definition [Bini-Capizzano, LAA03]

The GLT Space is the smallest closed algebra with respect to $d_\infty \times d_\infty$ that contains

$$\{T_n(f)\}_n \sim_{GLT} f(\theta) \quad \{D_n(a)\}_n \sim_{GLT} a(x) \quad Z_n \sim_{GLT} 0$$

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GLT Algebra [Serra-Capizzano, LAA03]

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Three Classes of Matrices

Examples of Symbol

- $Z_n \sim_{GLT} 0$
- $\{D_n(a)\}_n \sim_{GLT} a(x)$ where $a(x) \in C([0, 1])$
- $\{T_n(f)\}_n \sim_{GLT} f(\theta)$ where $f(\theta) \in L^1[-\pi, \pi]$

They appear frequently in PDEs

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0, 1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$A_n = D_n(a) T_n(2 - 2 \cos(\theta)) + Z_n$$

- The sequence $\{A_n\}_n$ has a spectral symbol?
- How do we compute it?

Three Classes of Matrices

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$$\{A_n\}_n \sim_{GLT} a(x)(2 - 2 \cos(\theta))$$

GLT properties

$$\widehat{\mathcal{E}} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n \times n}\} \quad \mathcal{M}_D = \{k : D \rightarrow \mathbb{C}, k \text{ measurable } \}$$

$$\begin{array}{ccc} \widehat{\mathcal{E}} & & \mathcal{M}_D \\ \cup \sqcup & & \cup \sqcup \\ P_1(\widehat{\mathcal{G}}) & & P_2(\widehat{\mathcal{G}}) \end{array}$$

Main Properties

1. $\widehat{\mathcal{G}}$ is **an algebra**
2. $\widehat{\mathcal{G}}$ is closed as a pseudometric space into $\widehat{\mathcal{E}} \times \mathcal{M}_D$
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More?

Identification

Let $\{A_n\}_n, \{C_n\}_n \in P_1(\mathcal{G})$.

1. S homomorphism of algebras

$$\implies S(\{A_n\}_n - \{C_n\}_n) = S(\{A_n\}_n) - S(\{C_n\}_n) = k_A - k_C$$

4. $\{A_n\}_n \sim_{\sigma} S(\{A_n\}_n)$

$$\implies \{A_n\}_n - \{C_n\}_n \sim_{\sigma} k_A - k_C$$

Th2. $\{A_n\}_n \sim_{\sigma} f \implies d_{acs}(\{A_n\}_n, \{0_n\}_n) = d_m(f, 0)$

$$\begin{aligned}\implies d_{acs}(\{A_n\}_n, \{C_n\}_n) &= d_{acs}(\{A_n\}_n - \{C_n\}_n, \{0_n\}_n) \\ &= d_m(k_A - k_C, 0) = d_m(k_A, k_C)\end{aligned}$$

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Let $k \in \mathcal{M}_D$

Let $k \in \mathcal{M}_D$ and $k_m \xrightarrow{\mu} k$ such that exist $S(\{B_{n,m}\}) = k_m$

Iso. S is an isometry

$\implies d_{acs}(\{B_{n,s}\}, \{B_{n,r}\}) = d_m(k_s, k_r) \implies \{B_{n,m}\}$ Cauchy

Th1. \mathcal{E} is complete $\implies \exists \{A_n\}_n : \{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$

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More?

We know that, for GLT, $\widetilde{\text{Im}}(S)$ is dense in \mathcal{M}_D , so

$$\mathcal{G} \cong \mathcal{M}_D$$

[Barbarino, LAA17]

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1. S is a homomorphism of groups
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$$\{A_n\}$$

- given $\{A_n\}_n$
- find $\{B_{n,m}\}_{n,m}$ GLT sequences with symbols k_m
- if k_m converges, then also $\{B_{n,m}\}_{n,m}$ converges
- if $\{B_{n,m}\}_{n,m}$ converges to $\{A_n\}_n$
- Then $\{A_n\}_n$ has spectral symbol k

→ proving the acs convergence is difficult

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Metrics on \mathcal{M}_D

Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be an increasing bounded concave and continuous function with $\varphi(0) = 0$

We can define corresponding metrics on \mathcal{E} and \mathcal{M}_D

$$p_m^\varphi(f) := \frac{1}{|D|} \int_D \varphi(|f|) \quad p^\varphi(\{A_n\}_n) := \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \varphi(\sigma_i(A_n))$$
$$d_m^\varphi(f, g) := p_m^\varphi(f - g) \quad d^\varphi(\{A_n\}_n, \{B_n\}_n) := p^\varphi(\{A_n - B_n\}_n)$$

Properties [Barbarino, Garoni, '17]

d^φ is a complete metric on \mathcal{E} inducing the a.s. convergence

$$\{A_n\}_n \sim_\sigma f \implies p^\varphi(\{A_n\}_n) = p^\varphi(f)$$

$$\{A_n\}_n \sim_{\text{var}} k, \{B_n\}_n \sim_{\text{var}} h \implies d^\varphi(\{A_n\}_n, \{B_n\}_n) = d^\varphi(k, h)$$

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Theorem 3 [Barbarino, Garoni, '17]

d^φ is a complete metric on \mathcal{E} inducing the acs convergence.

$$\{A_n\}_n \sim_\sigma f \implies p^\varphi(\{A_n\}_n) = p_m^\varphi(f)$$

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We can define corresponding metrics on \mathcal{E} and \mathcal{M}_D

$$p_m^\varphi(f) := \frac{1}{|D|} \int_D \varphi(|f|) \quad p^\varphi(\{A_n\}_n) := \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \varphi(\sigma_i(A_n))$$
$$d_m^\varphi(f, g) := p_m^\varphi(f - g) \quad d^\varphi(\{A_n\}_n, \{B_n\}_n) := p^\varphi(\{A_n - B_n\}_n)$$

Theorem 3 [Barbarino, Garoni, '17]

d^φ is a complete metric on \mathcal{E} inducing the acs convergence.

$$\{A_n\}_n \sim_\sigma f \implies p^\varphi(\{A_n\}_n) = p_m^\varphi(f)$$

$$\{A_n\}_n \sim_{GLT} k, \{B_n\}_n \sim_{GLT} h \implies d^\varphi(\{A_n\}_n, \{B_n\}_n) = d_m^\varphi(k, h)$$

Metrics on \mathcal{M}_D

Concave functions

- $\varphi_1(x) = \min\{x, 1\}$
- $\varphi_2(x) = \frac{x}{x+1}$

$$d_1^\varphi(\{A_n\}_n, \{B_n\}_n) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \min\{\sigma_i(A_n - B_n), 1\}$$

$$d_2^\varphi(\{A_n\}_n, \{B_n\}_n) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\sigma_i(A_n - B_n)}{\sigma_i(A_n - B_n) + 1}$$

→ New ways to test the a.s. convergence

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→ New ways to test the acs convergence

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