Ordering the Eigenvalues: GLT Symbols and Spectral Measures

Barbarino Giovanni

Scuola Normale Superiore

Asymptotic Distribution

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right)$$

Is there a connection like

$$\text{Ordering of eigenvalues} \overset{?}{\longleftrightarrow} \text{Spectral Symbol}$$

"To get good answers, you need good questions"

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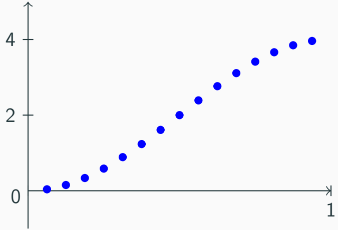

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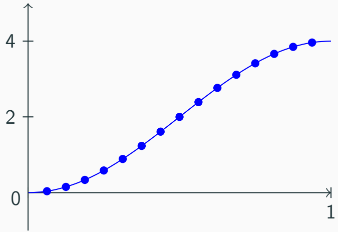

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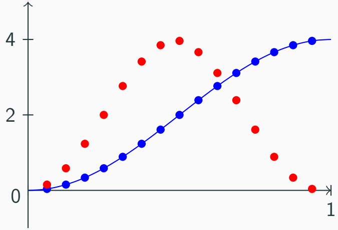

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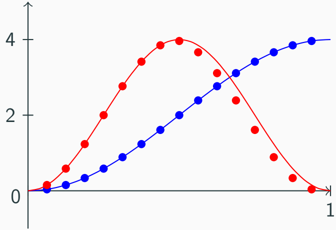

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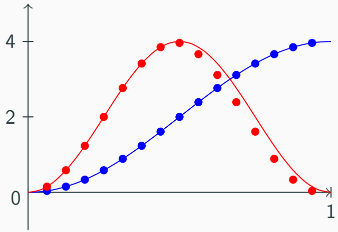

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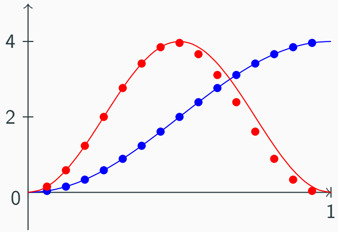

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Piecewise Convergence

Reduction

Spectral Symbol

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)}\int_D F(k(t))\mathrm{dt}$$

- $k(t)$ depends only on $\Lambda(A_n)$

$$\begin{array}{ccc} A_n & \rightsquigarrow & D_n := diag(\Lambda(A_n)) \\ \{A_n\}_n \sim_\lambda k & \Longrightarrow & \{D_n\}_n \sim_\lambda k \end{array}$$

We focus on

- Diagonal sequences $\{D_n\}_n$
- Spectral Symbols with domain $[0, 1]$

Warning: $A_n \rightsquigarrow D_n$ requires an **ordering choice** for $\Lambda(A_n)$

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Definition

$$D_n = \begin{bmatrix} d_1^{(n)} & & & & \\ & d_2^{(n)} & & & \\ & & d_3^{(n)} & & \\ & & & \ddots & \\ & & & & d_n^{(n)} \end{bmatrix}$$

Let $f_n : [0,1] \to \mathbb{C}$ be the piecewise linear function such that

$$f_n\left(\frac{i}{n}\right) = d_i^{(n)} \qquad \forall\, 1 \leq i \leq n$$

Piecewise Convergence

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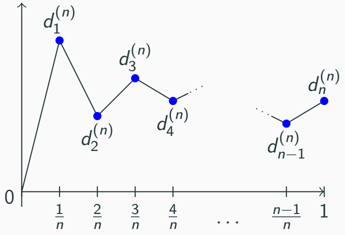

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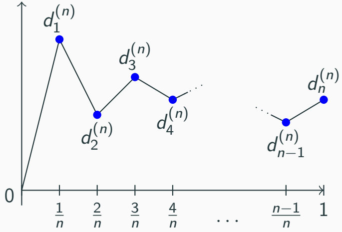

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Properties

- The piecewise convergence is linear

$$a, b \in \mathbb{C} \quad \{D_n\}_n \rightharpoonup k, \{D'_n\}_n \rightharpoonup h \implies \{aD_n + bD'_n\}_n \rightharpoonup ak + bh$$

- Zero distributed diagonal sequences converge piecewise to zero

$$\{Z_n\}_n \sim_\lambda 0 \iff \{Z_n\}_n \rightharpoonup 0$$

- Given $a \in C([0,1])$

$$\{D_n(a)\}_n \rightharpoonup a \qquad \{D_n(a)\}_n \sim_\lambda a$$

- For every $a : [0,1] \to \mathbb{C}$ there exists $\{D_n\}_n$ such that

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$$\{D_n\}_n \rightharpoonup f \implies \{D_n\}_n \sim_\lambda f$$

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We are actually interested in the inverse implication

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Rearrangement and GLT Symbol

Rearrangements

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)}\int_D F(k(t))\mathrm{dt}$$

$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < k(t) < b\}}{\mu(D)}$$

Rearrangement

Given $f : [0,1] \to \mathbb{R}$, then $g : [0,1] \to \mathbb{R}$ is a rearrangement if

$$\mu\{x : f(x) > z\} = \mu\{x : g(x) > z\} \qquad \forall z \in \mathbb{R}$$

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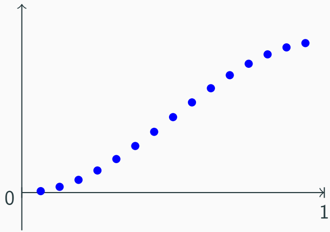

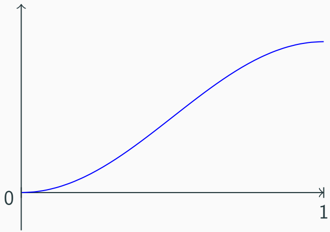

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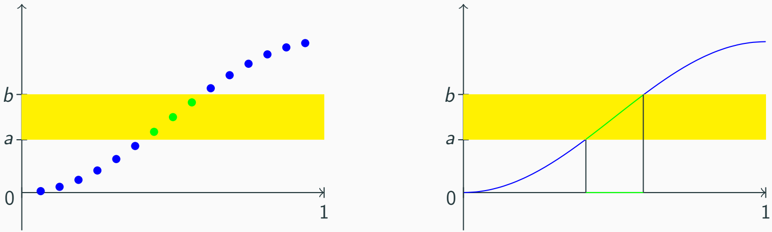

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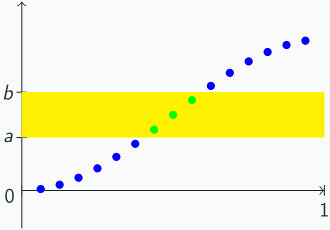

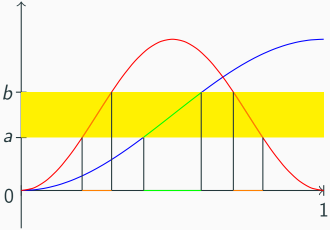

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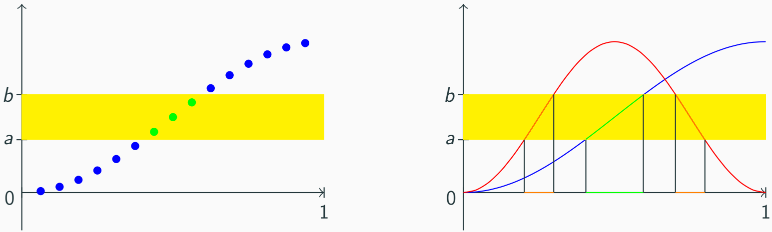

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Given $f : [0,1] \to \mathbb{R}$, then $g : [0,1] \to \mathbb{R}$ is a rearrangement if

$$\mu\{x : f(x) > z\} = \mu\{x : g(x) > z\} \qquad \forall z \in \mathbb{R}$$

Rearrangements

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t))\mathrm{dt}$$

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Main Properties

- If $\{A_n\}_n \sim_\lambda f$, and $f, g : [0,1] \to \mathbb{R}$, then

$$\{A_n\}_n \sim_\lambda g \iff g \text{ rearrangement of } f$$

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Key Lemma

Remember $\{D_n\}_n \rightharpoonup f \implies \{D_n\}_n \sim_\lambda f$

Key Lemma

Let D_n be real diagonal matrices, with decreasing entries. If $f : [0,1] \to \mathbb{R}$ is decreasing, then

$$\{D_n\}_n \sim_\lambda f \implies \{D_n\}_n \rightharpoonup f$$

Consequence

Let $\{D_n\}_n \sim_\lambda f$ real sequence and function.

g d.r. of f	$\{D_n\}_n \sim_\lambda g$
Decreasing sort	$\{P_n D_n P_n^T\}_n \sim_\lambda g$
Using Key Lemma	$\{P_n D_n P_n^T\}_n \rightharpoonup g$

The **choice of an order** is the same as the **choice of a symbol**?

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Main result

Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n real diagonal matrices and $f : [0,1] \to \mathbb{R}$, there exist P_n permutation matrices such that

$$\{P_n D_n P_n^T\}_n \rightharpoonup f(x)$$

Idea

Find $\{D'_n\}_n$ such that $\{D'_n\}_n \rightharpoonup f \quad \{D'_n\}_n \sim_\lambda f$

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Properties pt.2

- The piecewise convergence is linear

$$a, b \in \mathbb{C} \quad \{A_n\}_n \rightharpoonup k, \{B_n\}_n \rightharpoonup h \implies \{aA_n + bB_n\}_n \rightharpoonup ak + bh$$

- Zero distributed diagonal sequences converge piecewise to zero

$$\{Z_n\}_n \sim_\lambda 0 \iff \{Z_n\}_n \rightharpoonup 0 \impliedby \{Z_n\}_n \sim_{GLT} 0$$

- Given $a \in C([0,1])$

$$\{D_n(a)\}_n \rightharpoonup a \qquad \{D_n(a)\}_n \sim_\lambda a \qquad \{D_n(a)\}_n \sim_{GLT} a(x)$$

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GLT and Piecewise Convergence

Theorem

$$\{D_n\}_n \rightharpoonup f \iff \{D_n\}_n \sim_{GLT} f$$

Proof

Let $\{D_n'\}_n$ be a diagonal sequence such that

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The GLT symbols correspond to the **orderings of eigenvalues**

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The **GLT** **symbols** correspond to the **orderings of eigenvalues**

More?

Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n real diagonal matrices and $f : [0,1] \to \mathbb{R}$, then there exist P_n permutation matrices such that

$$\{P_n D_n P_n^T\}_n \sim_{GLT} f(x)$$

Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n complex diagonal matrices and $f : [0,1] \to \mathbb{C}$, then there exist P_n permutation matrices such that

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Warning: We used the decreasing rearrangement on $\mathbb{R}$

More?

Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n real diagonal matrices and $f : [0,1] \to \mathbb{R}$, then there exist P_n permutation matrices such that

$$\{P_n D_n P_n^T\}_n \sim_{GLT} f(x)$$

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Spectral Measures

Spectral Distribution

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)}\int_D F(k(t))\mathrm{dt}$$

$$\{A_n\}_n \sim_\lambda \phi : C_c(\mathbb{C}) \to \mathbb{C} \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \phi(F)$$

- ϕ is linear
- ϕ is continuous
- ϕ is positive

$$\implies \qquad \phi \in C_c(\mathbb{C})^*$$

Riesz Theorem

There exists an unique Radon measure μ such that

$$\phi(F) = \int_{\mathbb{C}} F\, d\mu \qquad \forall F \in C_c(\mathbb{C})$$

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and in this case, we write

$$\{A_n\}_n \sim_\lambda \mu$$

Probability Measure

Given $\{A_n\}_n \in \widehat{\mathscr{E}}$ the following are equivalent

- $\exists k : [0,1] \to \mathbb{C}$ s.t. $\{A_n\}_n \sim_\lambda k$
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Focus on $\mathbb{P}(\mathbb{C})$

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Vague Convergence

Given $\mu_n, \nu \in \mathbb{P}(\mathbb{C})$, then $\mu_n \xrightarrow{vague} \nu$ if

$$\int_{\mathbb{C}} F \, d\mu_n \to \int_{\mathbb{C}} F \, d\nu \qquad \forall F \in C_c(\mathbb{C})$$

Lévy-Prokhorov distance

The vague convergence is metrizable on $\mathbb{P}(\mathbb{C})$ through the distance
$\pi(\mu, \nu) = \inf \{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \ \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \ \forall A \in \mathscr{B}(\mathbb{C}) \}$

Properties

- If μ_n and μ correspond to functions f_n and f, then

$$f_n \xrightarrow{measure} f \implies \mu_n \xrightarrow{vague} \mu$$

- The Lévy-Prokhorov distance $\pi(\mu, \nu)$ is complete

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Atomic Measures

$$\{A_n\}_n \sim_\lambda \mu \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \int_{\mathbb{C}} F\, d\mu$$

- Given $A_n \in \mathbb{C}^{n\times n}$, we have

$$\mu_{A_n} := \frac{1}{n}\sum_{i=1}^{n}\delta(\lambda_i(A_n)) \implies \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \int_{\mathbb{C}} F\, d\mu_{A_n}$$

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$$\pi(A,\mu) := \pi(\mu_A,\mu) \qquad \pi(A,B) := \pi(\mu_A,\mu_B)$$

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The Spectral Measure of a sequence is **unique**

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Distance on Sequences

Lèvi-Prokhorov Distance on Sequences

$$\pi(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} \pi(A_n, B_n)$$

Theorem

π is a complete pseudometric on $\widehat{\mathscr{E}}$, and if $\mu, \nu \in \mathbb{P}(\mathbb{C})$ it respects

$$\{A_n\}_n \sim_\lambda \mu, \quad \{B_n\}_n \sim_\lambda \nu \implies \pi(\{A_n\}_n, \{B_n\}_n) = \pi(\mu, \nu)$$

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$$-\pi(\mu, A_n) + \pi(A_n, B_n) - \pi(B_n, \mu) \le \pi(\mu, \nu)$$

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$$\pi(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \pi(A_n, B_n) = \text{... too complicated}$$

We want a distance of similarity between the spectra

$$\Lambda(A) = \{\lambda_1(A), \dots, \lambda_n(A)\} \in \mathbb{C}^n$$

$$\Lambda(B) = \{\lambda_1(B), \dots, \lambda_n(B)\} \in \mathbb{C}^n$$

Optimal Matching Distance

$$d(v, w) = \min_{\sigma \in S_n} \max_{i=1,\dots,n} |v_i - w_{\sigma(i)}| \qquad d(A, B) := d(\Lambda(A), \Lambda(B))$$

$$\Lambda(A) = \Lambda(B) \iff d(A, B) = 0$$

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$$d(v, w) = \min_{\sigma \in S_n} \max_{i=1,\ldots,n} |v_i - w_{\sigma(i)}| \qquad d(A, B) := d(\Lambda(A), \Lambda(B))$$

$$\Lambda(A) = \Lambda(B) \iff d(A, B) = 0$$

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$$\pi(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \pi(A_n, B_n) = \text{... too complicated}$$

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Modified Optimal Matching Distance

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$$d(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} d(A_n, B_n)$$

Few outlier eigenvalues should not influence the distance

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$$d'(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_{\sigma\in S_n} \min_{i=1,\dots,n} \left\{ \frac{i-1}{n} + |\lambda(A_n) - \lambda_\sigma(B_n)|_i^{\downarrow} \right\}$$

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Equivalence of Measures

The space of sequences is endowed with pseudometrics π and d'

Theorem

$$\pi(\{A_n\}_n, \{B_n\}_n) \leq d'(\{A_n\}_n, \{B_n\}_n) \leq 2\pi(\{A_n\}_n, \{B_n\}_n)$$

They induce **the same topology**

Induced Properties

- d' is a complete pseudometric on $\widehat{\mathscr{E}}$
- If $\{A_n\}_n \rightsquigarrow \mu$, then

$$\{B_n\}_n \rightsquigarrow \mu \iff d'(\{A_n\}_n, \{B_n\}_n) = 0$$

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Closure Property

Let $\{B_{n,m}\}_n \sim_\lambda \mu_m$. Given

- $\pi(\mu_m, \mu) \to 0$
- $\{A_n\}_n \sim_\lambda \mu$
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two are true iff they are all true

Ideas

$$\pi(B_{n,m}, A_n) \le \pi(B_{n,m}, \mu_m) + \pi(\mu_m, \mu) + \pi(\mu, A_n)$$

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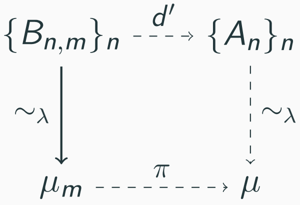

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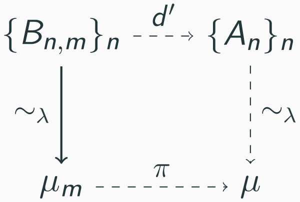

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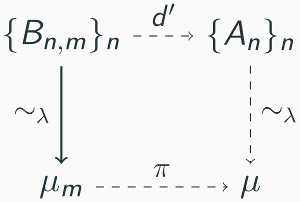

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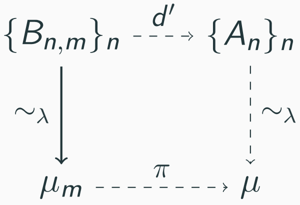

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Complex Case

Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n complex diagonal matrices and $f : [0,1] \to \mathbb{C}$, then there exist P_n permutation matrices such that

$$\{P_n D_n P_n^T\}_n \sim_{GLT} f$$

Idea

Find $\{D_n'\}_n$ such that

$$\{D_n'\}_n \sim_{GLT} f \qquad \{D_n'\}_n \sim_\lambda f$$

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References

- C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, vol. I. **Springer, 2017.**
- G. Barbarino, "Equivalence between GLT sequences and measurable functions," *Linear Algebra and its Applications* **529** (2017) 397–412.
- P. Tilli, "Locally Toeplitz sequences: spectral properties and applications," *Linear Algebra and its Applications* **278** no. 97, (1998) 91–120.
- S. Serra-Capizzano, "Distribution results on the algebra generated by Toeplitz sequences: a finite-dimensional approach," *Linear Algebra and its Applications* **328** no. 1-3, (2001) 121–130.
- S. Serra-Capizzano, "Generalized locally Toeplitz sequences: Spectral analysis and applications to discretized partial differential equations," *Linear Algebra and its Applications* **366** no. CORTONA 2000 Sp. Issue, (2003) 371–402.
- G. Barbarino, "Diagonal Matrix Sequences and their Spectral Symbols," `arXiv:1710.00810`. `http://arxiv.org/abs/1710.00810`.