Equivalence between GLT sequences and measurable functions

Barbarino Giovanni

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GLT sequences

Matrix Sequences

$$\widehat{\mathscr{E}} := \{\{A_n\}_n \mid A_n \in \mathbb{C}^{n\times n}\}$$

Approximating Class of Sequence [Serra-Capizzano, LAA01]

$\{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$ if

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}$$

for which exist $c(m), \omega(m), n_m$ such that

$$\frac{\operatorname{rk} R_{n,m}}{n} \le c(m) \qquad \|N_m\| \le \omega(m) \qquad \forall n > n_m$$

$$\lim_{n\to\infty} c(m) = \lim_{n\to\infty} \omega(m) = 0$$

→ The singular values of the difference tend to zero.

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$\rightarrow$ The singular values of the difference tend to zero.

Pseudometric Space

- The a.c.s. convergence is **metrizable**

$$d_{acs}\left(\{\{B_{n,m}\}_n\}_m, \{A_n\}_n\right) \to 0 \iff \{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$$

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} p_{acs}(A_n - B_n)$$

$$p_{acs}(A_n - B_n) := \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n - B_n) \right\}$$

corresponding to "small rank" R_n and "small norm" N_n

$$A_n - B_n = R_n + N_n$$

Theorem 1 [Barbarino, LAA17]
$(\widehat{\mathscr{E}}, d_{acs})$ is a complete pseudometric space.

Idea. Given a Cauchy sequence $\{\{B_{n,m}\}_n\}_m$, find a map $m(n)$ s.t.

$$\{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{B_{n,m(n)}\}_n$$

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Pseudometric Space

Let $\{B_{n,m}\}_{n,m}$ be a Cauchy sequence for d_{acs}

$B_{n,1}$	$B_{1,1}$	$B_{2,1}$	$B_{3,1}$	$B_{4,1}$	$B_{5,1}$	$B_{6,1}$	$B_{7,1}$	$B_{8,1}$	$B_{9,1}$
$d < \frac{1}{2}$		$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$
$B_{n,2}$	$B_{1,2}$	$B_{2,2}$	$B_{3,2}$	$B_{4,2}$	$B_{5,2}$	$B_{6,2}$	$B_{7,2}$	$B_{8,2}$	$B_{9,2}$
$d < \frac{1}{4}$				$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$
$B_{n,3}$	$B_{1,3}$	$B_{2,3}$	$B_{3,3}$	$B_{4,3}$	$B_{5,3}$	$B_{6,3}$	$B_{7,3}$	$B_{8,3}$	$B_{9,3}$
$d < \frac{1}{8}$					$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$
$B_{n,4}$	$B_{1,4}$	$B_{2,4}$	$B_{3,4}$	$B_{4,4}$	$B_{5,4}$	$B_{6,4}$	$B_{7,4}$	$B_{8,4}$	$B_{9,4}$
$d < \frac{1}{16}$							$p < \frac{1}{8}$	$p < \frac{1}{8}$	$p < \frac{1}{8}$
$B_{n,5}$	$B_{1,5}$	$B_{2,5}$	$B_{3,5}$	$B_{4,5}$	$B_{5,5}$	$B_{6,5}$	$B_{7,5}$	$B_{8,5}$	$B_{9,5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n									

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$d < \frac{1}{2}$		$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$
$B_{n,2}$	$B_{1,2}$	$B_{2,2}$	$B_{3,2}$	$B_{4,2}$	$B_{5,2}$	$B_{6,2}$	$B_{7,2}$	$B_{8,2}$	$B_{9,2}$
$d < \frac{1}{4}$				$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$
$B_{n,3}$	$B_{1,3}$	$B_{2,3}$	$B_{3,3}$	$B_{4,3}$	$B_{5,3}$	$B_{6,3}$	$B_{7,3}$	$B_{8,3}$	$B_{9,3}$
$d < \frac{1}{8}$					$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$
$B_{n,4}$	$B_{1,4}$	$B_{2,4}$	$B_{3,4}$	$B_{4,4}$	$B_{5,4}$	$B_{6,4}$	$B_{7,4}$	$B_{8,4}$	$B_{9,4}$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
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Pseudometric Space

$$d_{acs}(\{B_{n,m+1}\}_{n,m+1}, \{B_{n,m}\}_{n,m}) = \limsup_{n\to\infty} p_{acs}(B_{n,m+1} - B_{n,m})$$

$B_{n,1}$	$B_{1,1}$	$B_{2,1}$	$B_{3,1}$	$B_{4,1}$	$B_{5,1}$	$B_{6,1}$	$B_{7,1}$	$B_{8,1}$	$B_{9,1}$
$d < \frac{1}{2}$		$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$	$p < 1$
$B_{n,2}$	$B_{1,2}$	$B_{2,2}$	$B_{3,2}$	$B_{4,2}$	$B_{5,2}$	$B_{6,2}$	$B_{7,2}$	$B_{8,2}$	$B_{9,2}$
$d < \frac{1}{4}$				$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$	$p < \frac{1}{2}$
$B_{n,3}$	$B_{1,3}$	$B_{2,3}$	$B_{3,3}$	$B_{4,3}$	$B_{5,3}$	$B_{6,3}$	$B_{7,3}$	$B_{8,3}$	$B_{9,3}$
$d < \frac{1}{8}$					$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$	$p < \frac{1}{4}$
$B_{n,4}$	$B_{1,4}$	$B_{2,4}$	$B_{3,4}$	$B_{4,4}$	$B_{5,4}$	$B_{6,4}$	$B_{7,4}$	$B_{8,4}$	$B_{9,4}$
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$B_{n,2}$	$B_{1,2}$	$B_{2,2}$	$B_{3,2}$	$B_{4,2}$	$B_{5,2}$	$B_{6,2}$	$B_{7,2}$	$B_{8,2}$	$B_{9,2}$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	$B_{1,1}$	$B_{2,1}$	$B_{3,1}$	$B_{4,2}$	$B_{5,3}$	$B_{6,3}$	$B_{7,4}$	$B_{8,4}$	$B_{9,5}$

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Asymptotic Distribution

Spectral Symbol

Let $\{A_n\}_n \in \widehat{\mathscr{E}}$ and $k : D \to \mathbb{C}$ measurable.

$$\{A_n\}_n \sim_\sigma k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{\mu(D)} \int_D F(|k(t)|)\mathrm{dt}$$

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t))\mathrm{dt}$$

for all $F \in C_c(\mathbb{C})$.

Asymptotic Distribution

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$$A_n = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}$$

$$\lambda_k(A_n) = 2\cos\left(\frac{k\pi}{n+1}\right)$$

$k(t) = 2\cos(t)$

Asymptotic Distribution

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)}\int_D F(k(t))\mathrm{dt}$$

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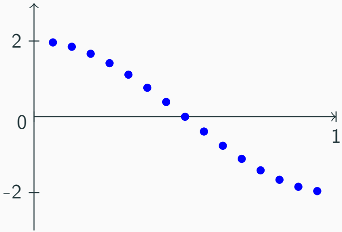

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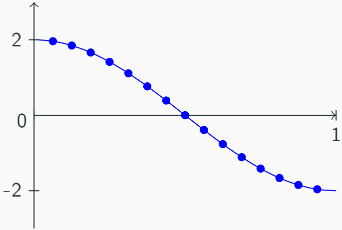

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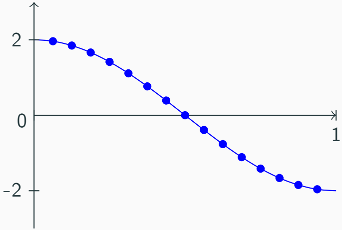

$$\lambda_k(A_n) = 2\cos\left(\frac{2k\pi}{n+1} + \left\lfloor\frac{2k}{n+1}\right\rfloor\frac{\pi}{n+1}\right) \qquad k(t) = 2\cos(t)$$

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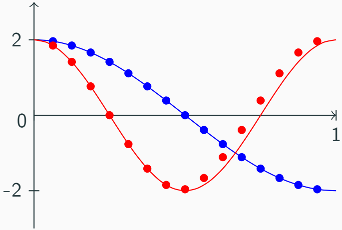

$$\lambda_k(A_n) = 2\cos\left(\frac{2k\pi}{n+1} + \left\lfloor \frac{2k}{n+1} \right\rfloor \frac{\pi}{n+1}\right) \qquad k(t) = 2\cos(2t)$$

Measurable Functions

$$\mathscr{M}_D = \{k : D \to \mathbb{C},\ k \text{ measurable } \}$$

- $\mathscr{M}_D$ is endowed with the convergence in **measure**: $k_m \xrightarrow{\mu} k$
- This convergence is induced by the complete pseudometric

$$d_m(f,g) = p_m(f-g) := \inf_{E \subseteq D} \left\{ \frac{|E^C|}{|D|} + \operatorname*{ess\,sup}_E |f-g| \right\}$$

Closure Property

$$\begin{array}{ccc} \{B_{n,m}\} & \xrightarrow{\text{acs}} & \{A_n\} \\ \Big\downarrow {\scriptstyle \sim_\sigma} & & \\ k_m & \xrightarrow{\mu} & k \end{array}$$

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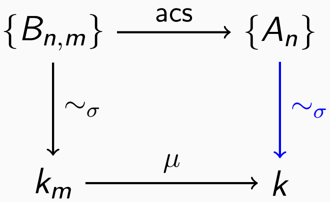
Closure Property



Correspondence of Metrics

Theorem 2 [Barbarino, LAA17]
If $\{A_n\}_n \sim_\sigma f$, then

$$d_{acs}(\{A_n\}_n, \{0_n\}_n) = \limsup_{n\to\infty} p_{acs}(A_n) = p_m(f) = d_m(f, 0)$$

Idea: $\{A_n\}_n$ $f(x)$

$$i \le j \implies \sigma_i \ge \sigma_j$$

$$\{\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1} \dots, \sigma_{n-1}, \sigma_n\}$$

$$p_{acs}(A_n) = \min_i \left\{ \frac{i-1}{n} + \sigma_i(A_n) \right\} \qquad p_m(f) = \inf_{z\in\mathbb{R}^+} \left\{ \frac{\mu(|f| > z)}{\mu(D)} + z \right\}$$

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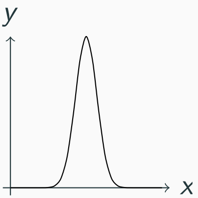

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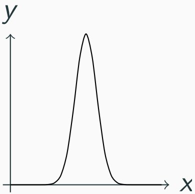

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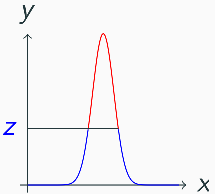

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GLT sequences

$$\widehat{\mathscr{G}} \subseteq \widehat{\mathscr{E}} \times \mathscr{M}_D$$

where $D = [0,1] \times [-\pi, \pi]$

- $\{T_n(f)\}_n \sim_{GLT,\sigma} f(\theta)$
- $\{D_n(a)\}_n \sim_{GLT,\sigma} a(x)$
- $Z_n \sim_{GLT,\sigma} 0$

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$$f \in L^1[-\pi,\pi] \to \widehat{f}_n = \int_{-\pi}^{\pi} f(\theta) e^{-\mathrm{i}n\theta} d\theta$$

$$T_n(f) := \begin{pmatrix} \widehat{f}_0 & \widehat{f}_1 & \widehat{f}_2 & \dots & \widehat{f}_{n-1} \\ \widehat{f}_{-1} & \widehat{f}_0 & \ddots & \ddots & \vdots \\ \widehat{f}_{-2} & \ddots & \ddots & \ddots & \widehat{f}_2 \\ \vdots & \ddots & \ddots & \widehat{f}_0 & \widehat{f}_1 \\ \widehat{f}_{-n+1} & \dots & \widehat{f}_{-2} & \widehat{f}_{-1} & \widehat{f}_0 \end{pmatrix}$$

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GLT Algebra [Serra-Capizzano, LAA03]

The GLT Space is the smallest **closed algebra** that contains

$$\{T_n(f)\}_n \sim_{GLT} f(\theta) \qquad \{D_n(a)\}_n \sim_{GLT} a(x) \qquad Z_n \sim_{GLT} 0$$

GLT properties

$\widehat{\mathscr{E}} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n\times n}\}$ $\qquad$ $\mathscr{M}_D = \{k : D \to \mathbb{C}, k \text{ measurable } \}$

$$\begin{array}{ccc} \widehat{\mathscr{E}} & \qquad & \mathscr{M}_D \\ \cup\!| & & \cup\!| \\ P_1(\widehat{\mathscr{G}}) & & P_2(\widehat{\mathscr{G}}) \end{array}$$

Main Properties

1. $\widehat{\mathscr{G}}$ is an algebra
2. $\widehat{\mathscr{G}}$ is closed as a pseudometric space into $\widehat{\mathscr{E}} \times \mathscr{M}_D$
3. GLT symbols are spectral symbols

   ($\widehat{\mathscr{G}}$ contains $\mathscr{Z}$ the set of zero-distributed sequences)

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More?

Identification

Let $\{A_n\}_n, \{C_n\}_n \in P_1(\mathscr{G})$.

1. S homomorphism of algebras

$$\implies S(\{A_n\}_n - \{C_n\}_n) = S(\{A_n\}_n) - S(\{C_n\}_n) = k_A - k_C$$

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Let $k \in \mathscr{M}_D$

Let $k \in \mathscr{M}_D$ and $k_m \xrightarrow{\mu} k$ such that exist $S(\{B_{n,m}\}) = k_m$

Iso. S is an isometry

$$\implies d_{acs}(\{B_{n,s}\}, \{B_{n,r}\}) = d_m(k_s, k_r) \implies \{B_{n,m}\} \text{ Cauchy}$$

$Th1.$ $\mathscr{E}$ is complete $\implies \exists \{A_n\}_n : \{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$

2. The graph of S into $\mathscr{E} \times \mathscr{M}_D$ is closed $\implies S(\{A_n\}_n) = k$

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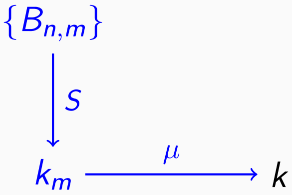

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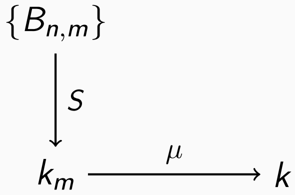

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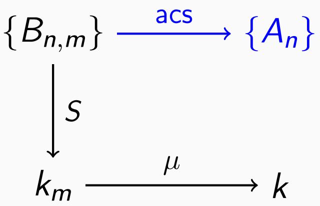

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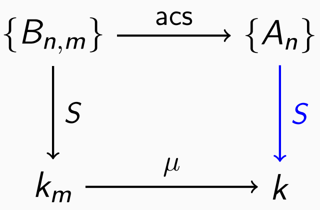

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More?

We know that, for GLT, $Im(S)$ is dense in $\mathscr{M}_D$, so

$\mathscr{G} \cong \mathscr{M}_D$ [Barbarino, LAA17]

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$$\mathscr{E} = \widehat{\mathscr{E}}/\mathscr{Z}$$
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Main Properties

1. S is a homomorphism of algebras
2. The graph of S into $\mathscr{E} \times \mathscr{M}_D$ is closed
3. $\{A_n\}_n \sim_\sigma S(\{A_n\}_n)$

S is an isometry and $Im(S)$ is closed

We know that, for GLT, $Im(S)$ is dense in $\mathscr{M}_D$, so

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$\{A_n\}$

- given $\{A_n\}_n$
- find $\{B_{n,m}\}_{n,m}$ GLT sequences with symbols k_m
- if k_m converges, then also $\{B_{n,m}\}_{n,m}$ converges
- if $\{B_{n,m}\}_{n,m}$ converges to $\{A_n\}_n$
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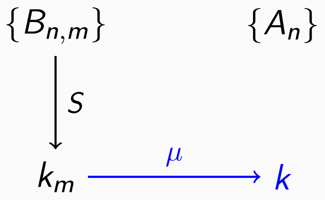

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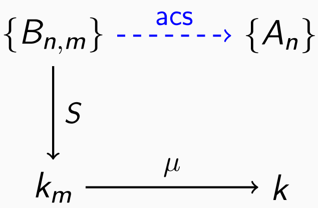

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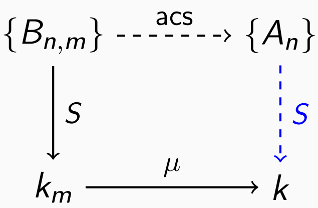

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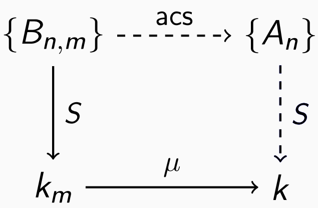

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Metrics on $\mathscr{M}_D$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be an increasing bounded concave and continuous function with $\varphi(0) = 0$

We can define corresponding metrics on $\mathscr{E}$ and $\mathscr{M}_D$

$$p_m^\varphi(f) := \frac{1}{|D|}\int_D \varphi(|f|) \qquad p^\varphi(\{A_n\}_n) := \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \varphi(\sigma_i(A_n))$$

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d^φ is a complete metric on $\mathscr{E}$ inducing the acs convergence.

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Concave functions

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$$d_1^{\varphi}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \min\{\sigma_i(A_n - B_n), 1\}$$

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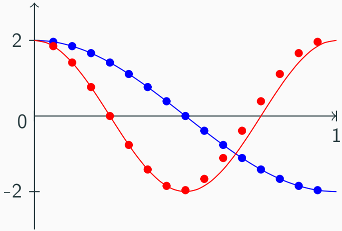
Distributions and Measures

Asymptotic Distribution

Spectral Symbol

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t))\mathrm{dt}$$

$$A_n = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}$$



Sorting Eigenvalues

Given $\{D_n\}_n \sim_\lambda f$ with D_n diagonal matrices and $f : [0,1] \to \mathbb{C}$, then there exist P_n permutation matrices such that

$$\{P_n D_n P_n^T\}_n \sim_{GLT} f(x)$$

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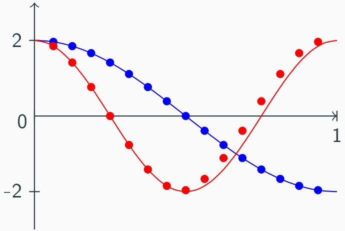

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Radon Measure

There exists a Radon Measure μ s.t.

$$\phi_k(F) = \int_{\mathbb{C}} F\, d\mu$$

so we can write

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Given $\{A_n\}_n \in \widehat{\mathscr{E}}$ the following are equivalent

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Vague Convergence

$$\mu_n \xrightarrow{\text{vague}} \mu$$

Properties

- Uniqueness of limit: $\{A_n\}_n \sim_\lambda \mu, \quad \{A_n\}_n \sim_\lambda \nu \implies \mu \equiv \nu$
- If μ_n and μ correspond to functions f_n and f, then

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Correspondence

Modified Optimal Matching

$$d'(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_{\sigma\in S_n} \min_{i=1,\dots,n} \left\{ \frac{i-1}{n} + |\lambda(A_n) - \lambda_\sigma(B_n)|_i^{\downarrow} \right\}$$

Closure Property

Let $\{A_{n,m}\}_n \sim_\lambda \mu_m$. Given

- $\pi(\mu_m, \mu) \to 0$
- $\{A_n\}_n \sim_\lambda \mu$
- $\{A_{n,m}\}_n \xrightarrow{d'} \{A_n\}_n$

two are true iff they are all true

More..

The distance d' is complete, and

$$\{A_n\}_n \sim_\lambda \mu_A, \quad \{B_n\}_n \sim_\lambda \mu_B \implies$$

$$\pi(\mu_A, \mu_B) \le d'(\{A_n\}_n, \{B_n\}_n) \le 2\pi(\mu_A, \mu_B)$$

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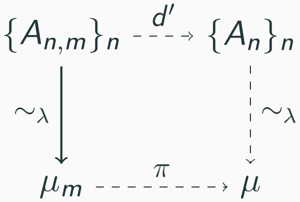

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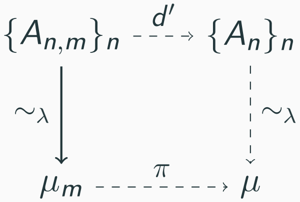

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