On the Rellich Eigendecomposition of Para-Hermitian Matrices on the Unit Circle

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MSC09 - Polynomial and rational matrices and applications
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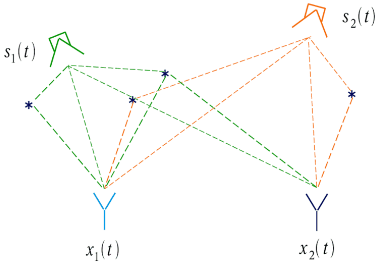
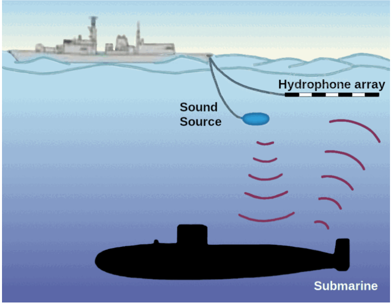
A first look to applications

Signal Decorrelation





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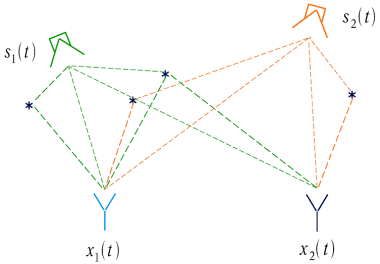

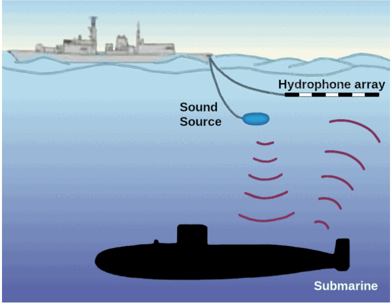

When the received signal $\{x_\tau\}_\tau$ is a convolutionary mixing of decorrelated signals, one can retrieve the original signal by diagonalizing the autocorrelation matrix of the z-series $x(z) = \sum_\tau x_\tau z^{-\tau}$ through Para-Unitary matrices

$$R(z) = \sum_\tau R_\tau z^{-\tau} \qquad R_\tau = \mathbb{E}[x_t x_{\tau - t}]$$

$$R(z) = Q(z)^{-1} \Sigma(z) Q(z)$$

The signal $Q(z)x(z)$ is now decorrelated since its autocorrelation matrix $\Sigma(z)$ is diagonal

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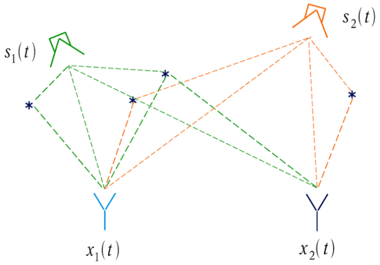

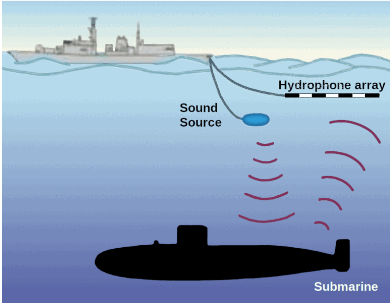

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$R(z)$ is a Para-Hermitian (PH) matrix polynomial:

$R(e^{i\theta})$ is Hermitian and $R_\tau^H = R_{-\tau}$

$Q(z)$ is Para-unitary (PU):

$Q(e^{i\theta})$ is unitary

How can we compute the **EVD** of a polynomial PH matrix?

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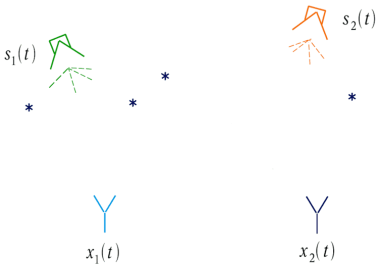

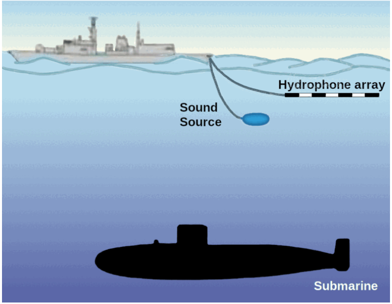

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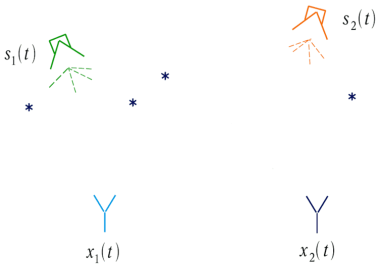

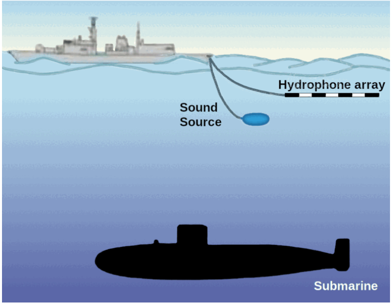

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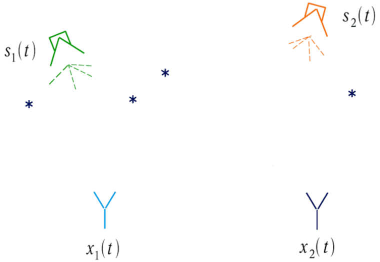

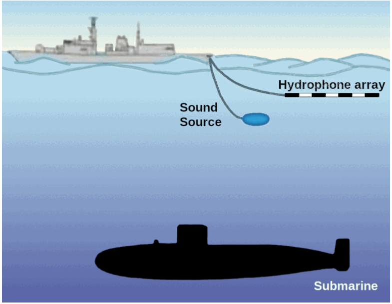

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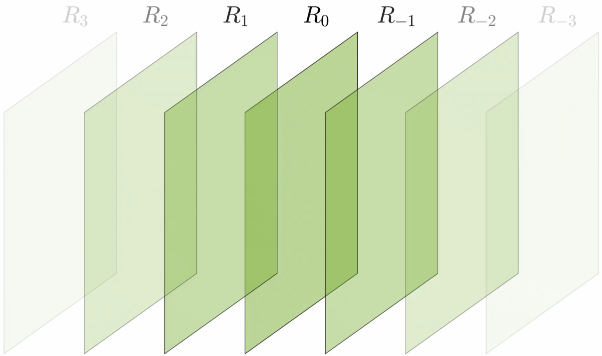
How can we compute the **EVD** of a polynomial PH matrix?

Second-Order Sequential Best Rotation : SBR2

$$R = \sum_\tau R_\tau z^{-\tau} = \sum_\tau \left[r_{i,j}^{(\tau)} \right]_{i,j} z^{-\tau} \qquad R_\tau^H = R_{-\tau}$$



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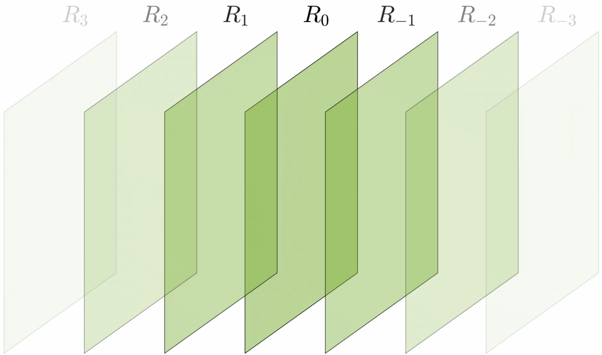
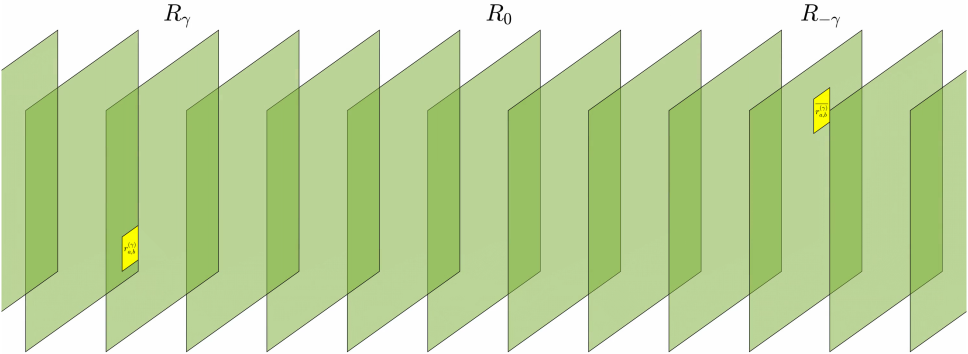


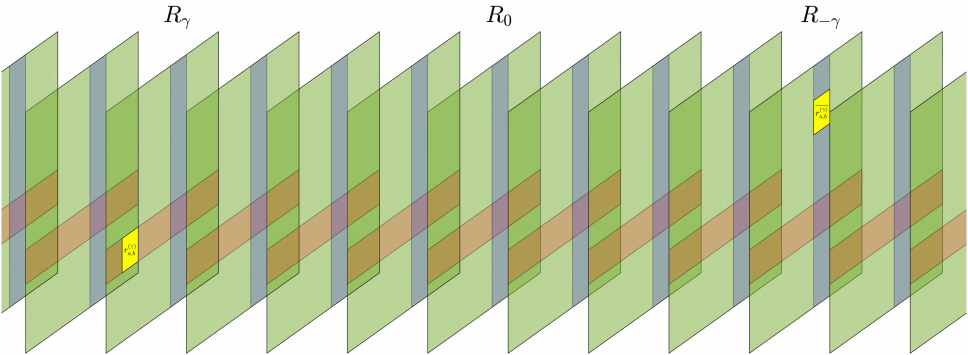
Signal Decorrelation

$$|r_{a,b}^{(\gamma)}| = \max_{i \neq j, \tau} |r_{i,j}^{(\tau)}| \qquad R_\tau = \left[r_{i,j}^{(\tau)} \right]_{i,j} \qquad r_{b,a}^{(-\gamma)} = \overline{r_{a,b}^{(\gamma)}}$$



Signal Decorrelation

$$\color{red}{diag(\ldots, 1, z^{\gamma}, 1, \ldots)}\color{black}{R(z)}\color{blue}{diag(\ldots, 1, z^{-\gamma}, 1, \ldots)}$$



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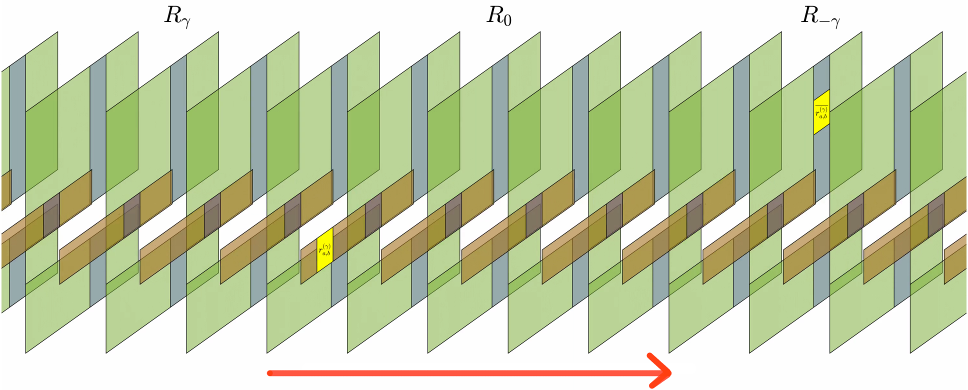

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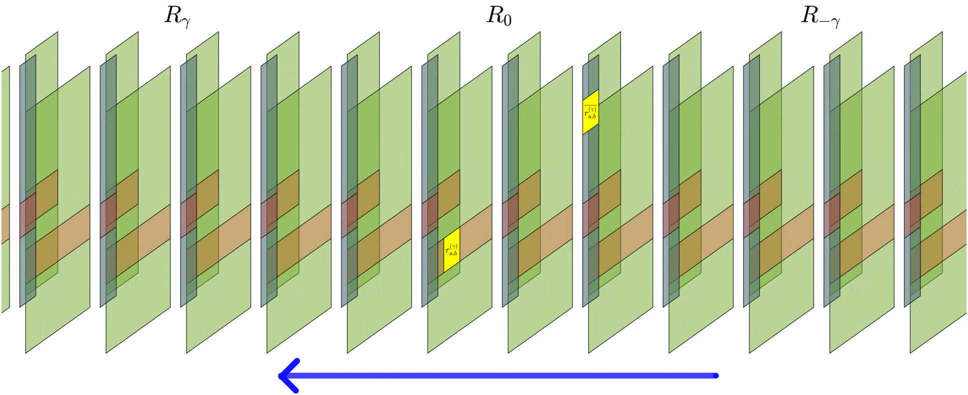

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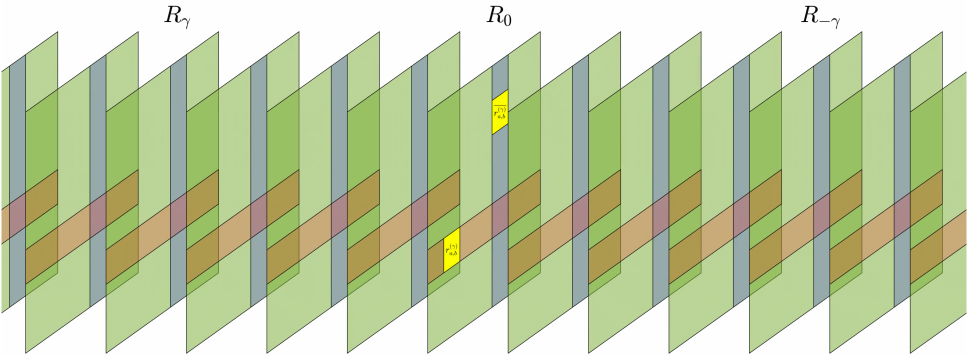
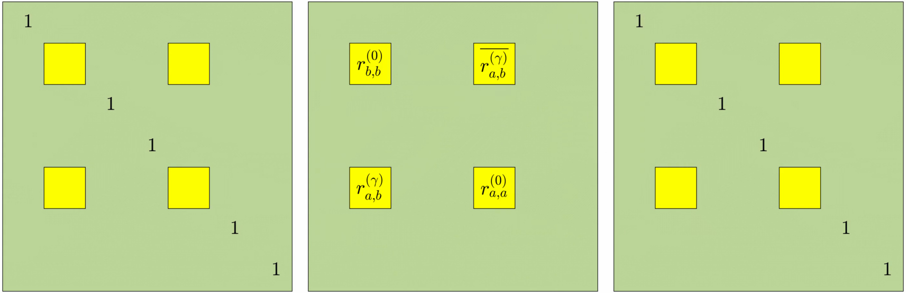


Signal Decorrelation

$$Q\, diag(\ldots, 1, z^{\gamma}, 1, \ldots) R(z) diag(\ldots, 1, z^{-\gamma}, 1, \ldots)\, Q^{-1}$$

$$R_0$$



Convergence of SBR2

The iterated steps of SBR2 are

- $|r_{a,b}^{(\gamma)}| = \max_{i \neq j, \tau} |r_{i,j}^{(\tau)}|$
- $QD_\gamma(z)R(z)D_{-\gamma}(z)Q^{-1} \to R(z)$

The invariant quantity $N := \sum_i |r_{i,i}^{(0)}|^2$ is bounded by the L^2 norm of all entries and for each step

$$N + 2|r_{a,b}^{(\gamma)}|^2 \to N$$

N thus converges and $|r_{a,b}^{(\gamma)}| \to 0$:
the off-diagonal entries converge uniformly to zero

The algorithm also converges for other metrics, such as the Coding Gain (PD case):

$$AM(diag\ R_0)\ /\ GM(diag\ R_0)$$

Problem

The multiplication by $D_\gamma(z)$ makes the degree of the polynomial rise by γ

$\to$ Number of off-diagonal elements rises

$\to$ More computationally expensive

Several variations and techniques addressing this problem have been developed: Trimming techniques, SMD, ME-SMD, AEVD, MSME-SMD, MS-SBR2, OCMS-SBR2, SBR2C
For all of them the convergence in norm is empirically observed but still missing

Conjecture

the L^2 norm of all off-diagonal elements tend to zero

$$\sum_{i \neq j} \|r_{i,j}(z)\|_{L^2}^2 = \sum_{i \neq j, \tau} \|r_{i,j}^{(\tau)}(z)\|^2$$

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Back to EVD

$$R(z) = U(z)\Sigma(z)U(z)^H$$

- $R(z)$ is PH and polynomial on S^1
- $U(z)$ is PU
- $\Sigma(z)$ is diagonal and real on S^1

SBR2 computes an EVD of $R(z)$, but its efficiency depends on the regularity of $U(z)$ and $\Sigma(z)$: non-smooth or non-holomorphic functions require high degree polynomials to be approximated.

Questions

→ Are there non-trivially different EVDs?

→ What are their regularity?

→ What EVD is the output of SBR2?

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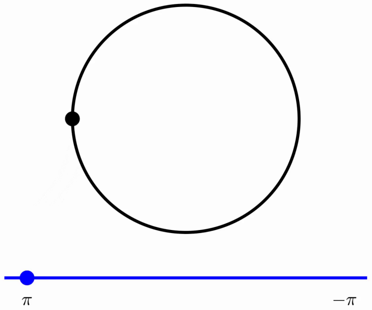
Analytic EVD

First Approach

$R(z)$ is holomorphic and Hermitian on S^1



$A(\theta) := R(e^{i\theta})$

analytic, periodic and Hermitian on R

Rellich Theorem

Given $A(\theta)$ analytical and Hermitian on an open interval $I \subseteq \mathbb{R}$, then it admits an analytical EVD on I

$$A(\theta) = Q(\theta)D(\theta)Q(\theta)^H$$

By the Fourier series of the real analytical EVD on $[-\pi, \pi]$ we obtain

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that is an holomorphic EVD on S^1 since $A(\theta)$ is periodic

....right?

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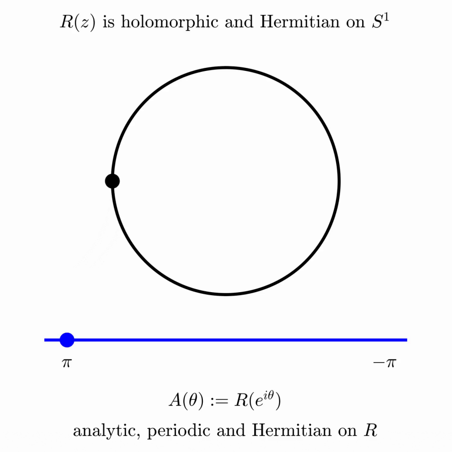

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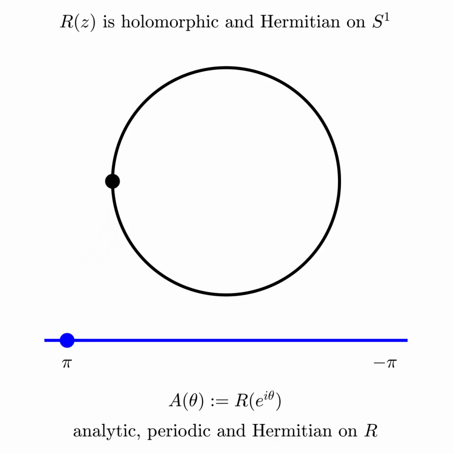

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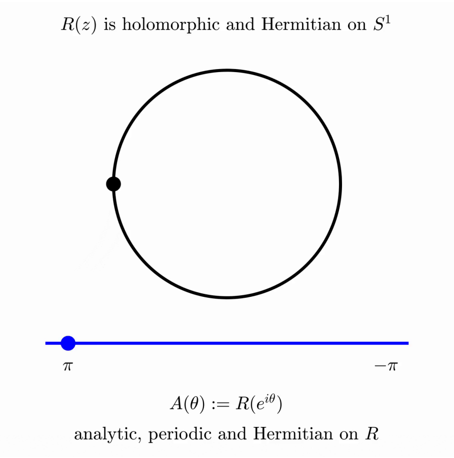

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Modulated Eigenvalues

$$R(z) = \begin{pmatrix} 0 & 1 - z^{-1} \\ 1 - z & 0 \end{pmatrix} \qquad A(\theta) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -ie^{i\theta/2} & ie^{i\theta/2} \end{pmatrix} \begin{pmatrix} 2\sin(\theta/2) & 0 \\ 0 & -2\sin(\theta/2) \end{pmatrix} \begin{pmatrix} 1 & ie^{-i\theta/2} \\ 1 & -ie^{-i\theta/2} \end{pmatrix}$$

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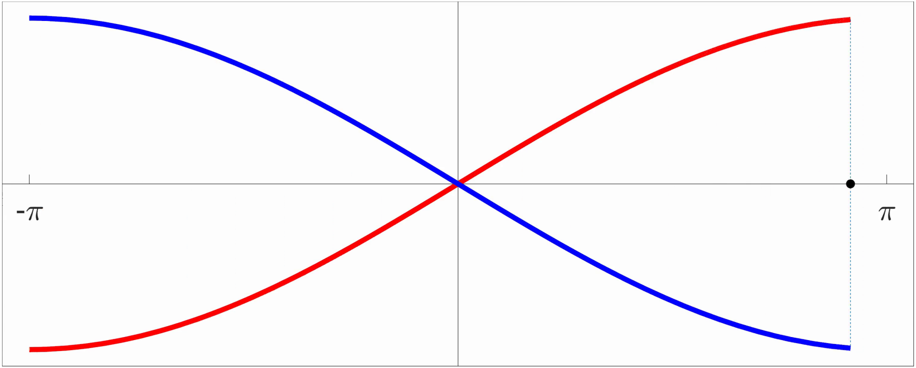

$$A(\theta) = \begin{pmatrix} 0 & 2+0.3i \\ 2-0.3i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1-0.1i & -1+0.1i \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1+0.1i \\ 1 & -1-0.1i \end{pmatrix}$$

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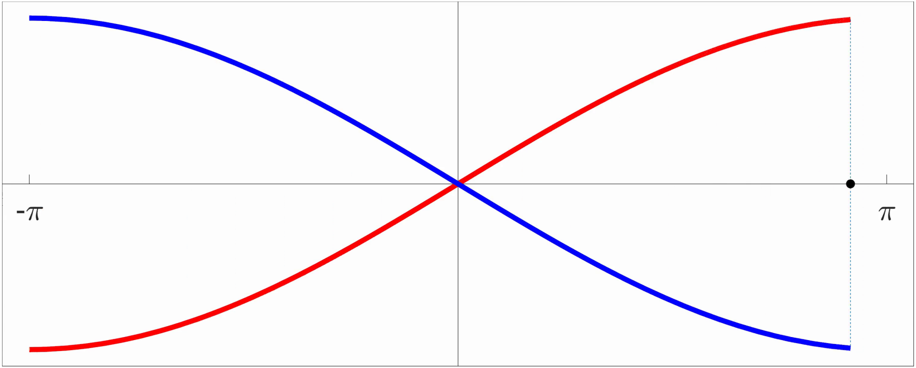

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They can come out from subband coders

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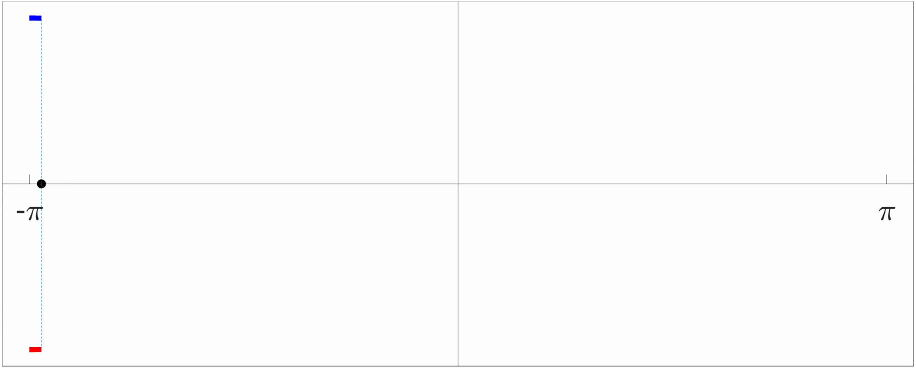

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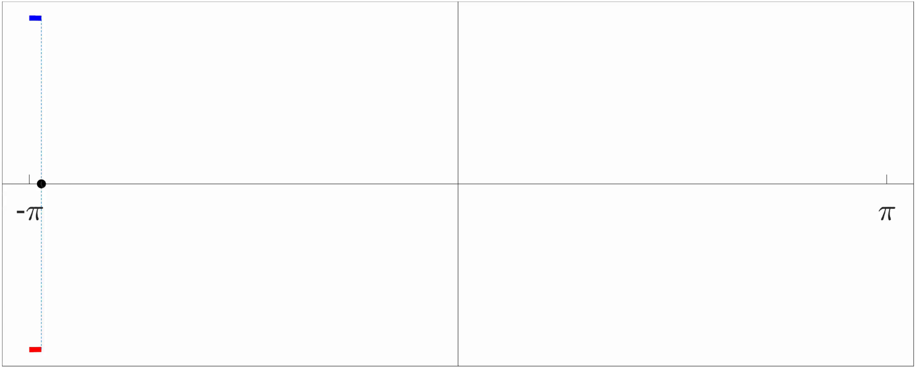

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They can come out from subband coders

Different typed of EVD

- There always exists an EVD $R(z) = U(z)\Sigma(z)U(z)^H$ with $\Sigma(z)$ and $U(z)$ holomorphic on $S^1/\{-1\}$, but in general not continuous on S^1
- There are examples of smooth PH $R(z)$ with non-modulated smooth eigenvalues but no continuous EVD
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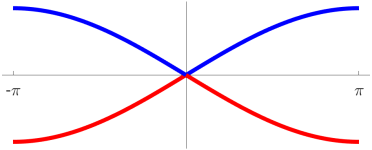

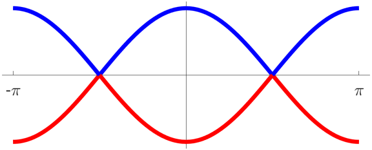

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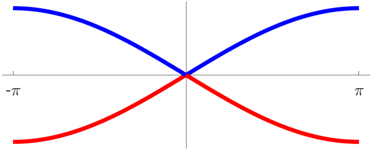

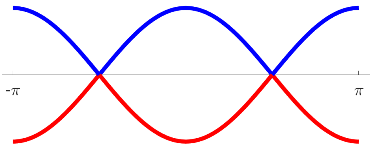

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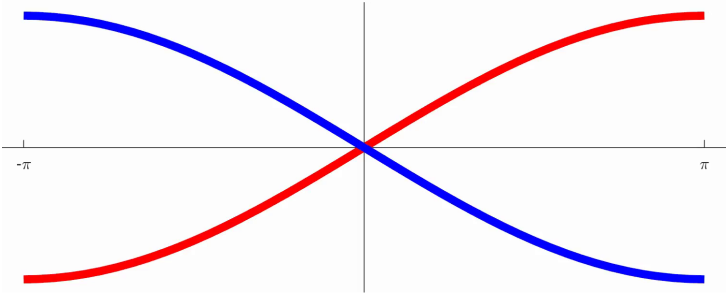
In these cases a DFT approach is preferred, more expensive but approximates the holomorphic solution if it exists

Puiseux Series

$$R(z) = \begin{bmatrix} 0 & 1 - z^{-1} \\ 1 - z & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -iz^{1/2} & iz^{1/2} \end{bmatrix} \begin{bmatrix} z^{1/2} - z^{-1/2} & 0 \\ 0 & z^{-1/2} - z^{1/2} \end{bmatrix} \begin{bmatrix} 1 & iz^{-1/2} \\ 1 & -iz^{-1/2} \end{bmatrix}$$



skip

Idea: In $A(\theta) = Q(\theta)D(\theta)Q(\theta)^H$, the eigenvalues of $D(-\pi)$ and $D(\pi)$ are just permuted, so if L is the order of the permutation, then $D(\theta)$ and $D(\theta + 2\pi L)$ have the same eigenvalues:

$R(z^L)$ has holomorphic eigenvalues

Theorem [B., Noferini (2023)]

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Equivalently, $R(z)$ admits an holomorphic EVD in the field of Puiseux series (holomorphic in $w = z^{1/L}$)

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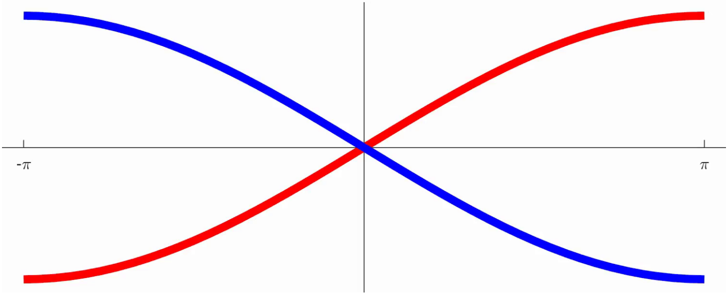

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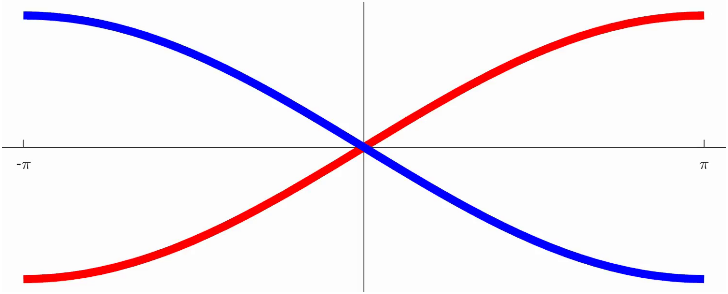

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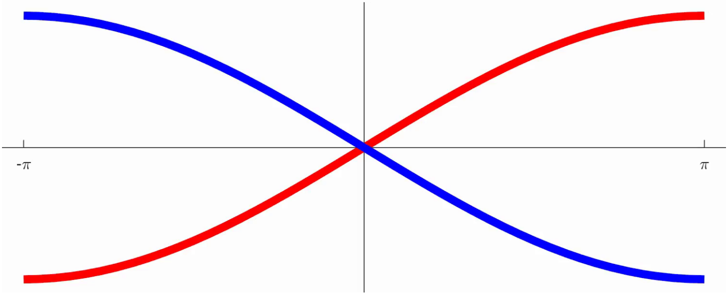

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Remarks and Consequences

Proof

$\Sigma(z)$ is holomorphic by construction

Theorem [Wimmer (1986)]

Given $A(\theta)$ analytical and Hermitian on an open interval $I \subseteq \mathbb{R}$, if it admits analytical eigenvalues on I, then it admits an analytical EVD on I.

- Wimmer only uses that the ring $\mathcal{H}(I)$ of holomorphic functions on I is an EDD (admits Smith Normal Form) and that $z \mapsto \overline{z}$ is in $\mathcal{H}(I)$
- The hypotheses are true for any I connected subset of the complex plane that are either lines or a circles
- The same can be proved if $R(z)$ is a matrix in Puiseux series

$z \mapsto \overline{z}$ is not holomorphic on any open subset of $\mathbb{C}$, but if I is a line or a circle on the complex plane, then it extends to a Moebius transformation and viceversa:

- Given a generic line $I = \{te^{i\theta} + \beta : t \in \mathbb{R}\}$,

$$\overline{z}|_I = e^{-2i\theta} z + \overline{\beta} - \beta e^{-2i\theta}$$

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Qeustion

Is there any non-trivial connected I that is not (subset of) a circle or a line, such that the conjugation is in $\mathcal{H}(I)$?

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$R(z^L)$ admits an holomorphic S^1 for any holomorphic (Puiseux) $n \times n$ PH matrix $R(z)$, where L is at most the Landau number $L(n) \sim \exp(\sqrt{n \log(n)})$, but n is usually small in applications (one can take $L = lcm(1, \dots, n) \sim e^n$)

Pseudo-Circulant: There exist $\phi_0(z), \phi_1(z), \dots, \phi_n(z) \in \mathcal{H}(S^1)$ for which $\overline{\phi_k(z)} = z^{-1}\phi_{n-k}$ and

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Its eigenvalues are modulated: there exists $\lambda(z) \in \mathcal{H}(S^1)$ such that $\lambda_j(e^{i\theta}) = \lambda(e^{2\pi ji/n}e^{i\theta/n})$

Theorem [B., Noferini (2023)]

Any holomorphic PH matrix $R(z)$ admits an holomorphic decomposition $R(z) = U(z)C(z)U(z)^H$ where $U(z)$ is PU, $C(z)$ is block diagonal with pseudo-circulant blocks and the block sizes reflect the periodicity of the eigenvalues

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Polynomial SVD

Theorem [B., Noferini (2023)]

Any holomorphic (Puiseux) rectangular matrix $M(z)$ admits an holomorphic SVD

$$M(z^L) = U(z)S(z)V(z)^H$$

for some integer L, where $U(z)$ and $V(z)$ are PU and $S(z)$ is rectangular, real and diagonal

$N(z) = \begin{pmatrix} 0 & M(z) \\ M(z)^H & 0 \end{pmatrix}$ is holomorphic and PH, so $N(z^L)$ admits a holomorphic EVD as

$$N(z^L) = \begin{pmatrix} 0 & M(z^L) \\ M(z^L)^H & 0 \end{pmatrix} = \begin{pmatrix} F(z) & F(z) \\ E(z) & -E(z) \end{pmatrix} \begin{pmatrix} \Lambda(z) & 0 \\ 0 & \Lambda(z) \end{pmatrix} \begin{pmatrix} F(z)^H & E(z)^H \\ F(z)^H & -E(z)^H \end{pmatrix}$$

so that the SVD becomes

$$M(z^L) = \sqrt{2}F(z) \cdot \Lambda(z) \cdot \sqrt{2}E(z)^H$$

Example:

$$[1 + z^2] = [z] \cdot [z + z^{-1}] \cdot [1] \qquad [1 + z] = [z^{1/2}] \cdot [z^{1/2} + z^{-1/2}] \cdot [1]$$

Polynomial SVD

Theorem [B., Noferini (2023)]

Any holomorphic (Puiseux) rectangular matrix $M(z)$ admits an holomorphic SVD

$$M(z^L) = U(z)S(z)V(z)^H$$

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Sign Characteristic

Definition

Given $f(x)$ analytic on I real interval with zeros x_i, let

$$f(x) = \epsilon_i c_i (x - x_i)^{m_i} + O(|x - x_i|^{m_i+1})$$

with $\epsilon_i = \pm 1$, $c_i > 0$

The **Sign Feature** of x_i is ϵ_i if the molteplicity m_i is odd and 0 if m_i is even

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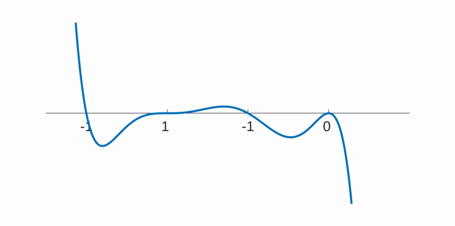

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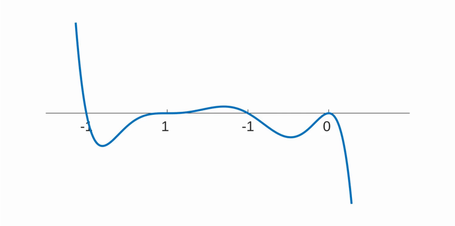

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The local sum of sign features is constant for small enough perturbations

Given $A(x)$ Hermitian analytic on I real interval with eigenvalues $\lambda_i(x)$ and finite eigenvalues x_j, notice that x_j are zeros of $det(A(x)) = \prod_i \lambda_i(x)$

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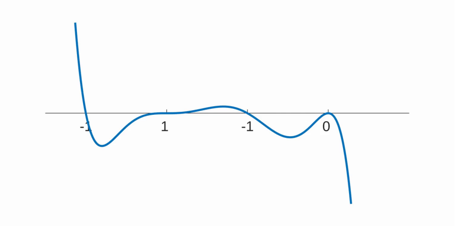

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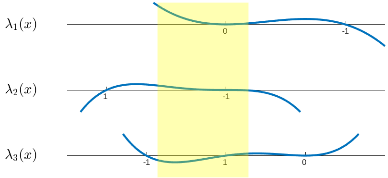

The local sum of sign features is constant for small enough Hermitian perturbations

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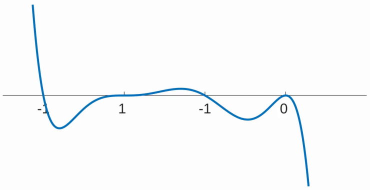

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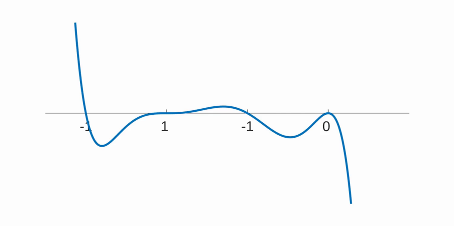

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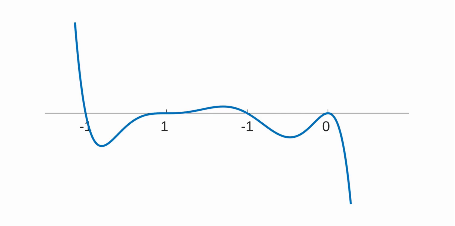

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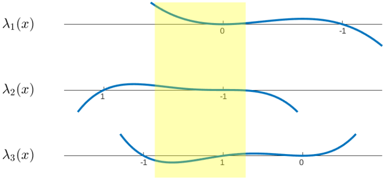

The local sum of sign features is constant for small enough Hermitian perturbations

Stability of Finite Eigenvalues

A finite eigenvalue is stable iff locally the sum of the sign features is not 0

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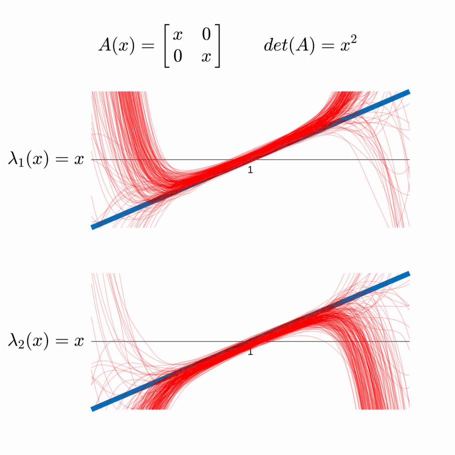

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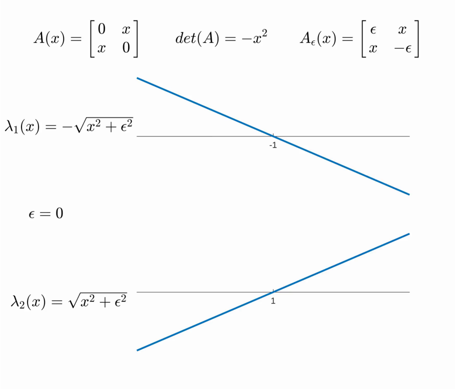

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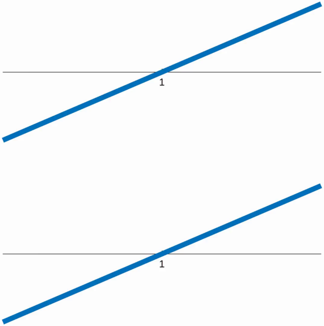
Stability of Finite Eigenvalues

A finite eigenvalue is stable iff locally the sum of the sign features is not 0

$$A(x) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \qquad det(A) = x^2$$

$\lambda_1(x) = x$

$\lambda_2(x) = x$

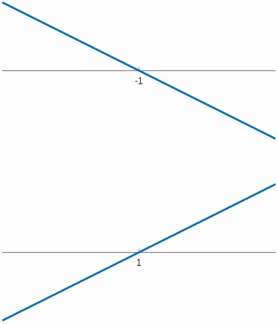


$$A(x) = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} \qquad det(A) = -x^2 \qquad A_\epsilon(x) = \begin{bmatrix} \epsilon & x \\ x & -\epsilon \end{bmatrix}$$

$\lambda_1(x) = -\sqrt{x^2 + \epsilon^2}$

$\epsilon = 0$

$\lambda_2(x) = \sqrt{x^2 + \epsilon^2}$



Palindromic Matrix Polynomials

$$P(z) = \sum_{i=0}^{g} P_i z^i \qquad P_{g-j} = P_j^H \qquad \Longrightarrow \qquad R(z) = z^{g/2} P(z) \quad \text{PH}$$

Let $\lambda_j(z)$ be the non-identically-zero eigenvalues of $R(z)$ in Puiseux series, and $z_k = e^{i\theta_k}$ the common finite eigenvalues of $P(z)$ and $R(z)$ on S^1

Then we can define the sign features of z_k as the sign features of θ_k with respect to $\lambda_j(e^{i\theta})$

Notice that changing the point where we rectify S^1 does not modify the local sum of sign features, up to the sign

The local sum of sign features is still constant for small enough palindromic perturbations

If $z_k = e^{i\theta_k}$ is a simple finite eigenvalue with eigenvector v, and $\det(R(z)) \not\equiv 0$, then its sign feature is equal to

$$\operatorname{sgn}\left[v^* \frac{dR(e^{i\theta})}{d\theta} v\right]_{\theta=\theta_k} = \operatorname{sgn}\left[i \frac{z_k}{z_k^{g/2}} \left[v^* \frac{dP(z)}{dz} v\right]_{z=z_k}\right]$$

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Conclusions and Future Works

Example 1:

$$P(z) = \begin{pmatrix} z+1 & i(z-1) \\ i(z-1) & 0 \end{pmatrix}$$

$$P_\epsilon(z) = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^2(z+1) \end{pmatrix}$$

The matrix $P(z) + P_\epsilon(z)$ has close finite eigenvalues

$$\lambda = \frac{(1 \pm i\epsilon)^2}{1+\epsilon^2}$$

with sign features ± 1, so they must be instable.

In fact, the matrix $P(z) - P_\epsilon(z)$ has finite eigenvalues

$$\lambda = \frac{1 \pm \epsilon}{1 \mp \epsilon}$$

that do not belong to S^1

Example 2:

$$Q(z) = \begin{pmatrix} i(z-1) & \gamma(z+1) \\ \gamma(z+1) & i(z-1) \end{pmatrix} = Bz + B^H$$

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For any matrix $\|A\| < 1$, the palindromic polynomial $Q_A(z) = Q(z) + Az + A^H$ still possesses two finite eigenvalues on S^1:

Using $z \in S^1/\{-1\} \iff w = \frac{1-z}{i(1+z)} \in \mathbb{R}$, we find that $Q_A(z)$ presents unimodular finite eigenvalues iff

$$i(B - B^H + A - A^H)w + (B + B^H + A + A^H)$$

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Thank You!

- Mackey D.S., Mackey N., Mehl C., and V. Mehrmann. **Structured polynomial eigenvalue problems: Good vibrations from good linearizations.** *SIAM Journal on Matrix Analysis and Applications*, 28:1029–1051, 2006.
- Rellich F. ***Perturbation Theory of Eigenvalue Problems*****.** 1969.
- Barbarino G. and Noferini V. **On the rellich eigendecomposition of para-hermitian matrices and the sign characteristics of *-palindromic matrix polynomials.** *Linear Algebra and its Applications*, 672:1–27, 2023.
- Wimmer H.K. **Rellich's perturbation theorem on hermitian matrices of holomorphic functions.** *Journal of Mathematical Analysis and Applications*, 144(1):52–54, 1986.
- Weiss S., Proudler I. K., Barbarino G., Pestana J., and McWhirter J. **Existence and uniqueness of the analytic singular value decomposition.** 2023.
- Weiss S., Proudler I. K., and Pestana J. **On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix.** *IEEE Transactions on Signal Processing*, 66(10):2659–2672, 2018.
- Mehrmann V., Noferini V., Tisseur F., and Xu H. **On the sign characteristics of hermitian matrix polynomials.** *Linear Algebra and its Applications*, 511(15):328–364, 2016.