Iterative Filtering Algorithms

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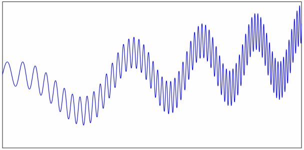
DI GEOFISICA E VULCANOLOGIA

19-20 May 2023

Iterative Filtering

Empirical Method Decomposition (EMD)

Signal $s(x)$



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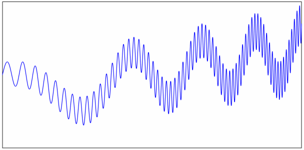

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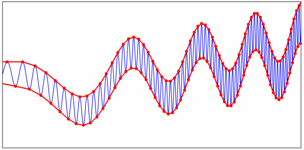

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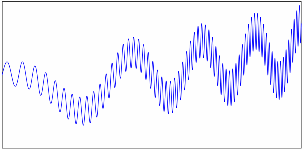
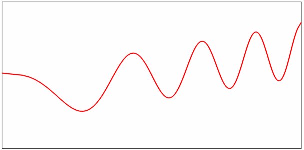
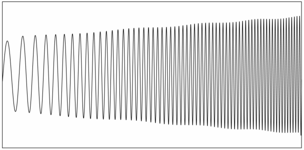


Moving Average $\mathcal{L}(s)$

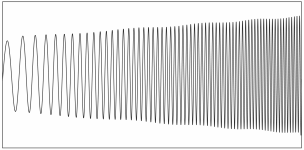


$\mathcal{S}(s) = s - \mathcal{L}(s)$

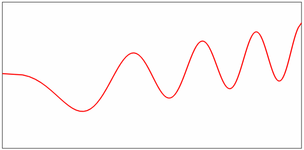


Empirical Method Decomposition (EMD)

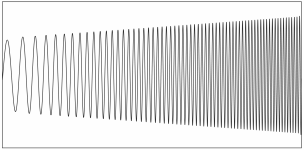
$\mathcal{S}(s)$



$\mathcal{L}(s) + \mathcal{L}(\mathcal{S}(s))$

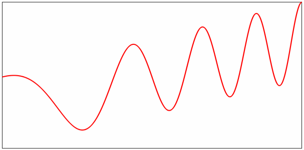


$\mathcal{S}^2(s)$

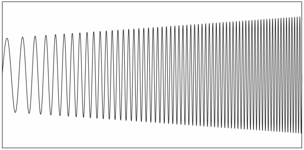


Empirical Method Decomposition (EMD)

Remainder Signal $s_1(x)$

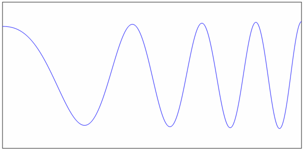


$IMF_1 = \mathcal{S}^{\infty}(s)$

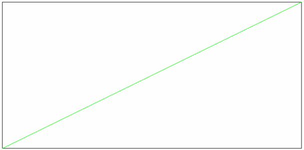


Empirical Method Decomposition (EMD)

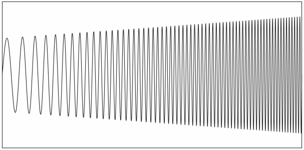
IMF_2



IMF_3



IMF_1



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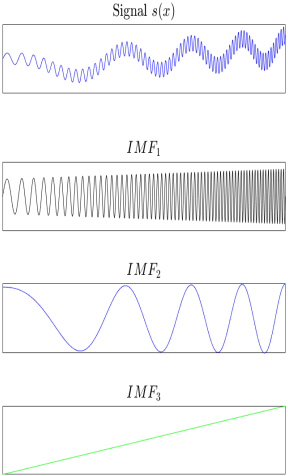


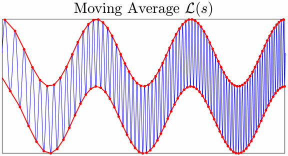
Decomposition of non-stationary signals into Intrinsic Mode Functions (IMF)

- Iterative Method
- Based on the computation of the moving average of the signal
- Splits the signal into simple oscillatory components

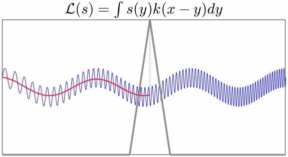
Numerous variants (EEMD, NA-MEMD, FMEMD, etc.) have been proposed in the years to deal with instability and mode splitting/mixing, and to prove its convergence

Iterative Filtering

The effect of the moving average is to flatten the highest frequency component



A way to emulate the effect is to use a filter on the signal



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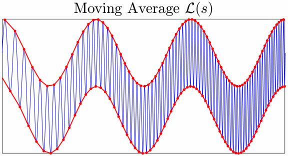

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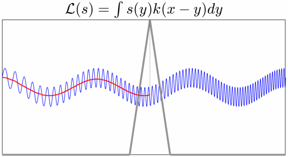

Choose the filter k:

- Unit-norm, even, nonnegative and compact supported
- $k = \omega \star \omega$

$$\implies 0 \leq \hat{k}(\xi) \leq 1$$

The IF method iteratively apply the filter through convolution

$$\mathcal{S}(f) := f(x) - \int f(y)k(x-y)dy$$
$$IMF = IMF \cup \{\mathcal{S}^\infty(s)\}$$
$$s = s - \mathcal{S}^\infty(s)$$

The convergence of $\mathcal{S}^\infty(s)$ can be studied on the frequencies space

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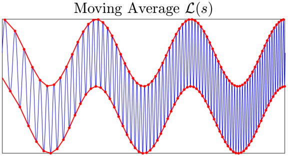

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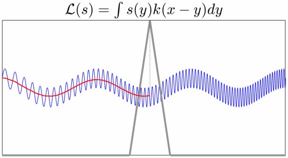

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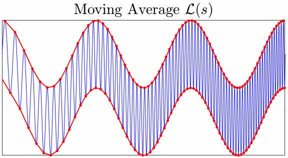

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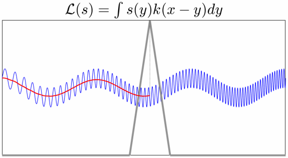

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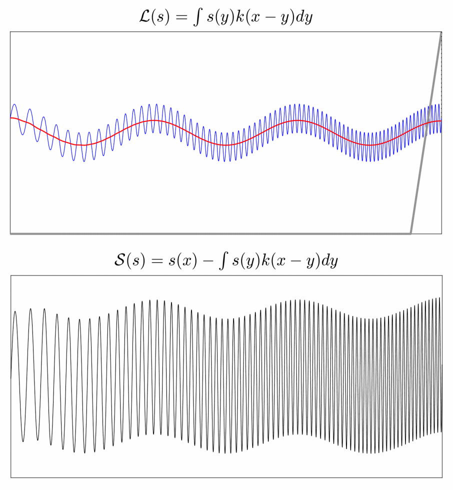
Time-Frequency Space

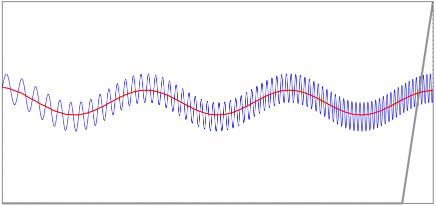
On the Time Dimension the Sifting Operator is the difference between the signal and the Moving Average



Time-Frequency Space

On the Time Dimension the Sifting Operator is the difference between the signal and the Moving Average

$\mathcal{L}(s) = \int s(y)k(x-y)dy$



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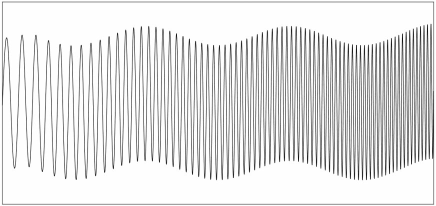
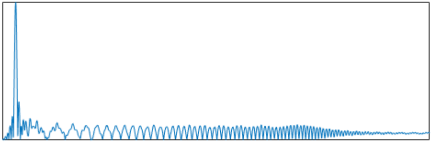


On the Frequency Dimension

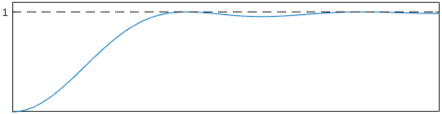
$$\widehat{\mathcal{S}(s)}(\xi) = \widehat{s}(\xi)(1-\widehat{k}(\xi))$$

$$\widehat{\mathcal{S}^m(s)}(\xi) = \widehat{s}(\xi)(1-\widehat{k}(\xi))^m$$

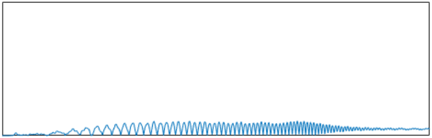
$|\widehat{s}(\xi)|$



$1-\widehat{k}(\xi)$



$|\widehat{\mathcal{S}(s)}(\xi)| = |\widehat{s}(\xi)|(1-\widehat{k}(\xi))$



The Fundamental Zero and the Stopping Criterion

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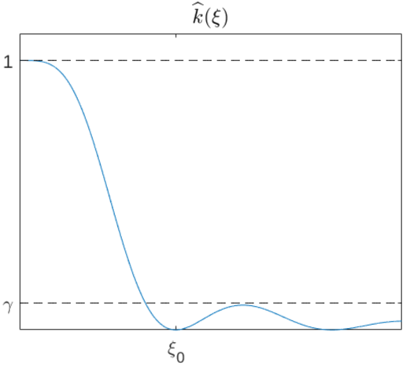
The Sifting Operator extracts the frequencies corresponding to low values of $0 \leq \widehat{k}(\xi)$



Call J_γ the neighbourhood of ξ_0 the first zero of $\widehat{k}(\xi)$ on which $\widehat{k} < \gamma$

$$|\widehat{\mathcal{S}^m(s)}(\xi)| \leq |\widehat{s}(\xi)|(1-\gamma)^m \quad \xi \notin J_\gamma$$

Notice that $Lk(Lx)$ is also a filter with ξ_0/L as first zero

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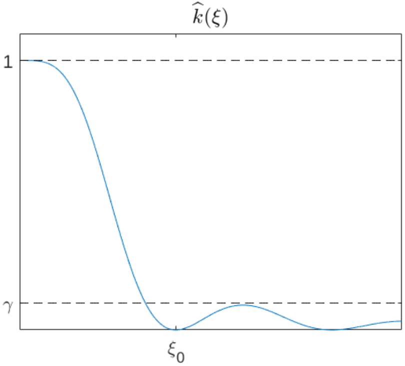

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Set the Stopping Criterion for IF as

$$\|\mathcal{S}^{m+1}(s) - \mathcal{S}^m(s)\| < \delta$$

Theorem (Cicone, Zhou, 2021)

Given $0 \le \widehat{k} \le 1$, $\delta > 0$, $s(x) \in L^2(\mathbb{R})$, *then*

$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|s\|}$$

implies $\|\mathcal{S}^{m+1}(s) - \mathcal{S}^m(s)\| < \delta$

If m is the stopping index, $m = O(\|s\|/\delta)$

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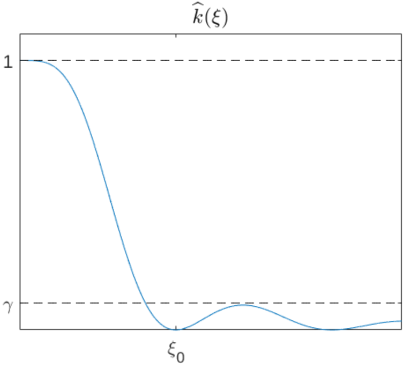

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If $I_\gamma := \{\xi : (1-\widehat{k}(\xi))^m > 1-\gamma\}$ *and*

$$\widehat{IMF}^{TH} = \chi_{I_\gamma}\widehat{s} + (1-\widehat{k})^m(1-\chi_{I_\gamma})\widehat{s},$$

then for any fixed $\eta > 0$*, there exist* $\delta, \gamma > 0$ *for which* $\|IMF - IMF^{TH}\| \le \eta/2$ *for all IMFs*

Order of the Fundamental Zero

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Better results are achieved with bigger I_γ, especially for amplitude-modulated signals where

$$s(x) = a(x)g(x) \implies \widehat{s}(\xi) = (\widehat{a} \star \widehat{g})(\xi)$$

and if $a(x)$ has low instant frequency, then $\widehat{a} \star \widehat{g}$ has non-zero components near the main frequencies of g

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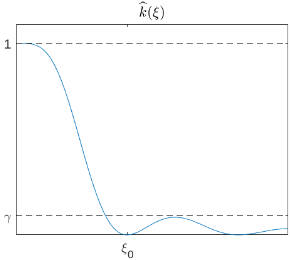

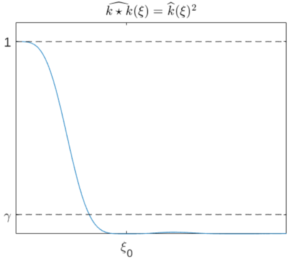

Better performances are also achieved with smoother filters and

$$k \star k \star k \star \dots$$

is a more regular filter with the same first zero of the FT

Theorem (B. 2023)

If we choose ξ_0 *depending on the biggest frequency in* $\widehat{s}$ *whose intensity is at least* η*, then*

$$B(\xi_0, C\sqrt[2p]{\eta\delta}) \subseteq I_\gamma$$

where $2p$ *is the order for the first zero in* $\widehat{k}$

Frequency Partition and Perturbation

Theorem (Cicone, Zhou 2021, B. 2023)

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$$\widehat{IMF}_j = \widehat{\mathcal{S}^{m_j}(r_j)} = \widehat{r_j}(1 - \widehat{k_j})^{m_j}$$

where r_j is what's left after having extracted $j-1$ IMFs from the original signal $s(x)$, so

$$\widehat{r_{j+1}} = \widehat{r_j} - \widehat{IMF}_j = \widehat{r_j}[1 - (1 - \widehat{k_j})^{m_j}]$$

Theorem (B. 2023)

$$\widehat{IMF}_j(\xi) = \lambda_j(\xi) \cdot \widehat{s}(\xi)$$

where $0 \le \lambda_j(\xi)$ *and* $\sum_j \lambda_j(\xi) \le 1 \quad \forall\xi$.
Thus, there is a finite number of relevant IMF, i.e. $\|IMF_j\| > \eta$

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Thus, there is a finite number of relevant IMF, i.e. $\|IMF_j\| > \eta$

This is important for perturbations, since

Theorem (B. 2023)

For any $h, s \in L^2$

$$\|\mathcal{S}^m(s+h) - \mathcal{S}^m(s)\| \leq \|h\|$$

and if we fix k_j, m_j *in the algorithm, for* IMF_j *the modes generated by* $s(x)$ *and for* IMF_j^* *generated by* $s(x)+h(x)$, *we have*

$$\sum_j \|IMF_j^* - IMF_j\|^2 \leq \|h\|^2.$$

Discrete Setting

The signal $s(x)$ is studied on $[0, 1]$ and it is supposed to be periodic at the boundaries [**Stallone, Cicone, Materassi 2020**] so that the discretization results in a circulant matrix

$$\boldsymbol{s} = [s(h)\ s(2h)\ \dots\ s(1-h)\ s(1)] \qquad h = 1/N$$

$$\mathcal{S}(s)(x) = s(x) - \int_0^1 s(x-y)k(y)dy|_{x=ah} \qquad \mathcal{S}(s)(ah) \sim \boldsymbol{s}_a - \frac{1}{N}\sum_{b=1}^{N} k\,(bh)\,\boldsymbol{s}_{a-b}$$

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One can thus write the main loop of the discrete IF Algorithm as

$\mathcal{S}(\boldsymbol{f}) := (I-K)\boldsymbol{f}$

$\boldsymbol{IMF} = \boldsymbol{IMF} \cup \{\mathcal{S}^m(\boldsymbol{s})\}$

$\boldsymbol{s} = \boldsymbol{s} - \mathcal{S}^m(\boldsymbol{s})$

where the stopping condition is $\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta$

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$$\mathcal{S}(\boldsymbol{s}) := \boldsymbol{s} - K\boldsymbol{s} = (I-K)\boldsymbol{s}$$

One can thus write the main loop of the discrete IF Algorithm as

$\mathcal{S}(\boldsymbol{f}) := (I-K)\boldsymbol{f}$

$\boldsymbol{IMF} = \boldsymbol{IMF} \cup \{\mathcal{S}^m(\boldsymbol{s})\}$

$\boldsymbol{s} = \boldsymbol{s} - \mathcal{S}^m(\boldsymbol{s})$

where the stopping condition is $\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta$

Fast IF

$$\mathcal{S}^m(\boldsymbol{s}) = (I-K)^m\boldsymbol{s} \implies \widehat{\mathcal{S}^m(\boldsymbol{s})} = \boldsymbol{k}^{\circ m} \circ \widehat{\boldsymbol{s}}$$

where $\boldsymbol{k}$ is the first row of $I-K$, $\circ$ is the elementwise product and $\widehat{\boldsymbol{s}}$ is the DFT of $\boldsymbol{s}$

$$\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta \iff \|\boldsymbol{k}^{\circ m} \circ (\boldsymbol{k} - \boldsymbol{e}) \circ \widehat{\boldsymbol{s}}\| < \delta$$

The stopping condition can be checked on $\boldsymbol{k}$ and $\widehat{\boldsymbol{s}}$ with linear cost + 2 DFT per IMF

Theorems in the Discrete Settings

$$\mathcal{S}^m(\boldsymbol{s}) = (I - K)^m \boldsymbol{s} \implies \widehat{\mathcal{S}^m(\boldsymbol{s})} = \boldsymbol{k}^{\circ m} \circ \widehat{\boldsymbol{s}}$$

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Theorem

If k is a filter, then $0 \leq \boldsymbol{k} \leq 1$, so $\mathcal{S}^m(\boldsymbol{s})$ always converges

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If k is a filter, then $0 \le \boldsymbol{k} \le 1$*, so* $\mathcal{S}^m(\boldsymbol{s})$ *always converges*

Theorem (Cicone, Zhou, 2021, B. 2023)

Given $\delta > 0$*,* $\boldsymbol{s}$*, then*

$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|\boldsymbol{s}\|}$$

implies $\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta$

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Theorem (B. 2023)

$$\widehat{\boldsymbol{IMF}}_j = \boldsymbol{\lambda}_j \circ \widehat{\boldsymbol{s}}$$

where $0 \leq \boldsymbol{\lambda}_j$ *and* $\sum_j \boldsymbol{\lambda}_j \leq 1$*. Thus, there is a finite number of relevant IMF, i.e.* $\|\boldsymbol{IMF}_j\| > \eta$

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For any vectors $\boldsymbol{h}, \boldsymbol{s}$ let K be any $n \times n$ Hermitian matrix with spectrum in $[0,1]$. Then

$$\|\mathcal{S}^m(\boldsymbol{s}+\boldsymbol{h}) - \mathcal{S}^m(\boldsymbol{s})\| \le \|\boldsymbol{h}\|.$$

If now the filters and m_j are fixed, for $\boldsymbol{IMF}_j$ the modes generated by $\boldsymbol{s}$ and for $\boldsymbol{IMF}_j^*$ generated by $\boldsymbol{s}+\boldsymbol{h}$, we have

$$\sum_j \|\boldsymbol{IMF}_j^* - \boldsymbol{IMF}_j\|^2 \le \|\boldsymbol{h}\|^2.$$

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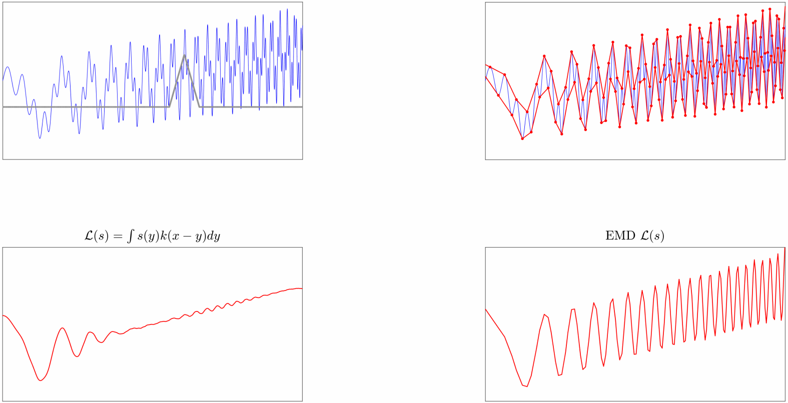
Theorem (B. 2023)

The approximation error of $\boldsymbol{IMF}_j$ with respect to the continuous algorithm modes IMF_j is proportional to $\log(1/\delta)/n$

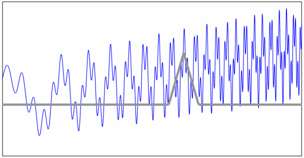
Drawbacks



Let's take a look at the instantaneous frequencies (don't skip)

⏩ skip

Drawbacks



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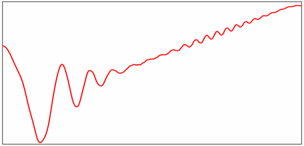
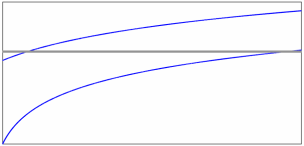


Instantaneous Frequencies

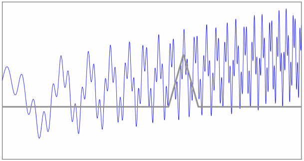


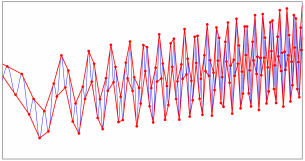
$$\widehat{\mathcal{S}(s)}(\xi) = \widehat{s}(\xi) \cdot (1 - \widehat{k}(\xi))$$

IF does not work with non-disjoint bands of frequencies

skip

Drawbacks





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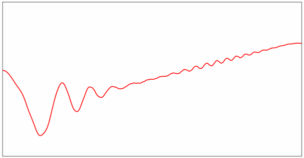
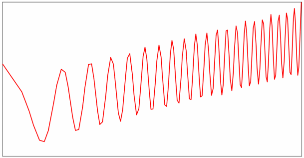
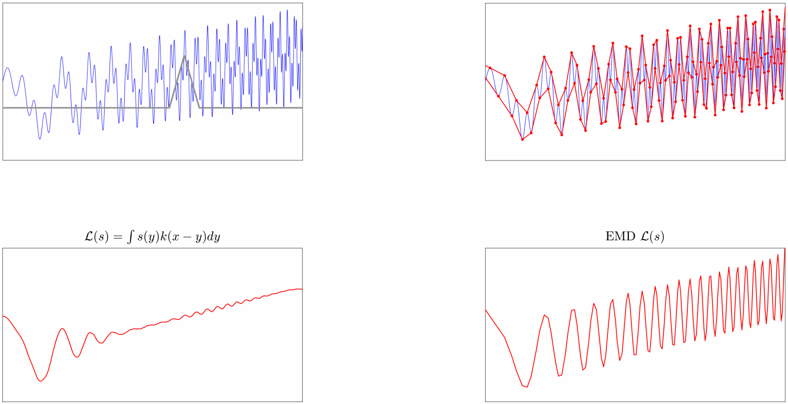


EMD $\mathcal{L}(s)$



Drawbacks



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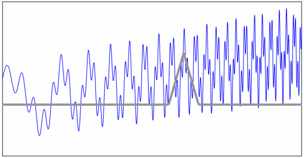
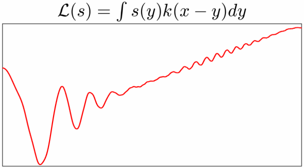
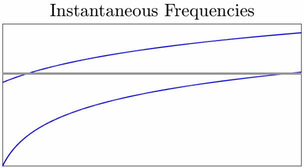
Drawbacks







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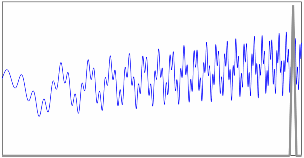
IF does not work with non-disjoint bands of frequencies

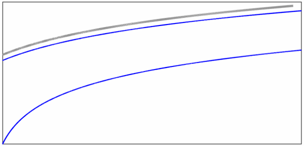
Adaptive Local Iterative Filtering

Adaptive Local Iterative Filtering

$$k_x(y) := k(\ell(x)^{-1}y)\ell(x)^{-1} \qquad \mathcal{S}(s)(x) := s(x) - \int s(y)k_x(x-y)dy$$



Instantaneous Frequencies



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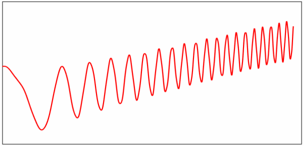

skip

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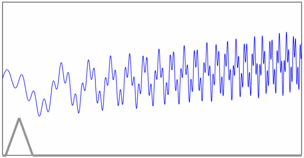

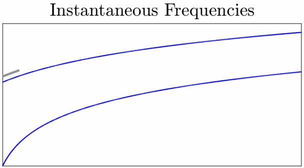

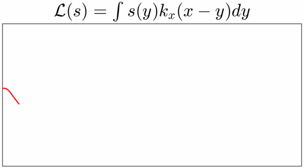

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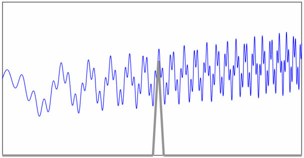

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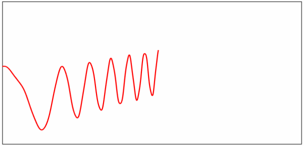
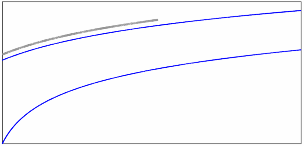


Instantaneous Frequencies



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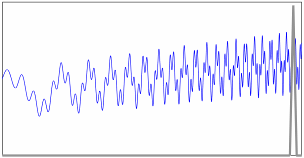

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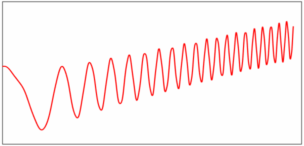

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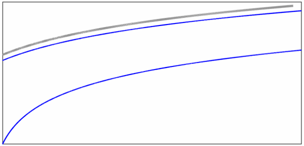
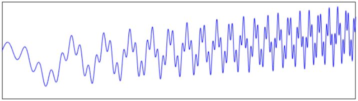


Adaptive Local Iterative Filtering

Signal $s(x)$



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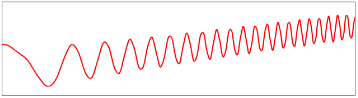

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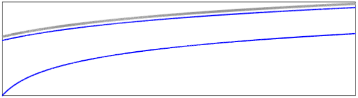

Given the signal $s(x)$, fix the filter

$$k_x(y) := k(\ell(x)^{-1}y)\ell(x)^{-1}$$

where ideally $\ell(x) \sim \xi_0/f(x)$, with $f(x)$ being the instantaneous frequency of the higher-frequency IMF.

Apply iteratively the filter through sifting

$$\mathcal{S}(f) := f(x) - \int f(y)k_x(x-y)dy$$
$$IMF = IMF \cup \{\mathcal{S}^\infty(s)\}$$
$$s = s - \mathcal{S}^\infty(s)$$

ALIF is now as flexible as EMD, and empirically converges, but..

- No structure, not fast as IF ($O(n^2)$ against $O(n\log(n))$)
- Has no clean formal analysis since it is not a convolution
- $\mathcal{S}^\infty(s)$ is not always convergent (in the discrete setting) even with a stopping condition

Discrete ALIF

$$\boldsymbol{s} = [s(h)\ s(2h)\ \dots\ s(1-h)\ s(1)] \qquad h = 1/N$$

$$s(x) - \int_0^1 s(y) k_x(x-y) dy|_{x=ah} \quad \sim \quad \boldsymbol{s}_a - \frac{1}{N} \sum_{b=1}^{N} k\left(\frac{(a-b)h}{\ell(ah)}\right) \frac{1}{\ell(ah)} \boldsymbol{s}_b$$

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- $\mathcal{S}^\infty(\boldsymbol{s})$ converges when

$$|\lambda_i(I-K)| < 1 \vee \lambda_i(I-K) = 1$$

- Converges to the kernel of K

The kernel is the same in αM where $\alpha \in \mathbb{R}$, so the real condition is

$$\Im(\lambda_i(K)) > 0 \vee \lambda_i(K) = 0$$

Setting a stopping condition in the iteration makes $\mathcal{S}^\infty(\boldsymbol{s})$ a near-kernel vector

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For big enough N and if $\ell(x)$ is continuous, positive and

$$k(x) = \omega(x) \star \omega(x),$$

then the spectrum of K respects the condition for almost every eigenvalue [**B., Cicone 2022**]

There are artificial examples where K has negative eigenvalues, so the convergence is not always assured

Stable ALIF

Given the ALIF matrix K, let

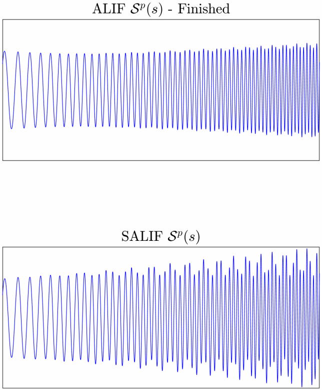
$$\mathcal{S}(\boldsymbol{s}) := \boldsymbol{s} - K^T K \boldsymbol{s} = (I - K^T K)\boldsymbol{s}$$

- $K^T K$ Has the same kernel of K
- $1 \geq \lambda_i(K^T K) \geq 0$ after a renormalization

As a consequence, $\mathcal{S}^\infty(\boldsymbol{s})$ always converges, but the method is **way slower**

- The cost per iteration is doubled
- There are more eigenvalues close to zero, so it takes more iterations to extract the exact component

$T \times 1/20$ $N = 3000$



skip

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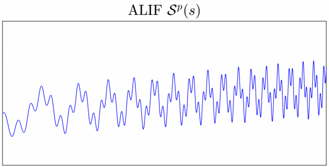

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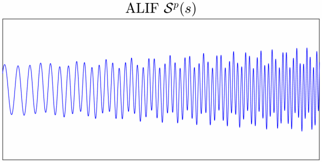

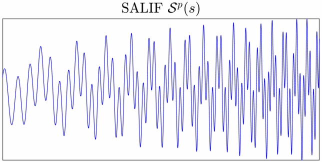

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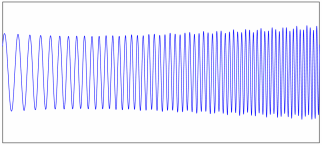
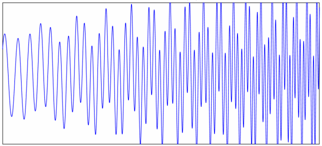
$T = 2/20$ $N = 3000$

ALIF $\mathcal{S}^p(s)$ - Finished



SALIF $\mathcal{S}^p(s)$



Stable ALIF

$T = 3/20$ $N = 3000$

Given the ALIF matrix K, let

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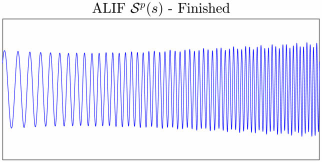

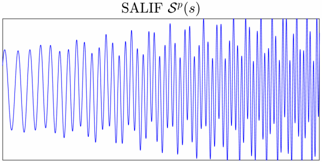

Stable ALIF

Given the ALIF matrix K, let

$$\mathcal{S}(\boldsymbol{s}) := \boldsymbol{s} - K^T K \boldsymbol{s} = (I - K^T K)\boldsymbol{s}$$

- $K^T K$ Has the same kernel of K
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As a consequence, $\mathcal{S}^\infty(\boldsymbol{s})$ always converges, but the method is **way slower**

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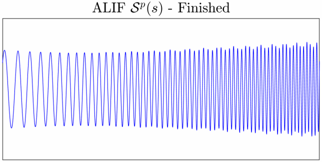

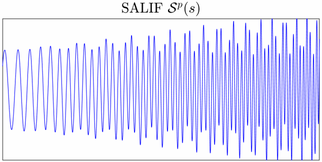

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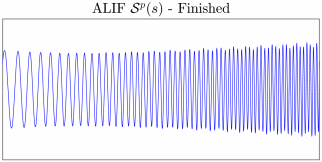

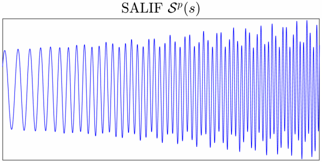

Results about SALIF

$$\mathcal{S}(\boldsymbol{s}) = (I - K^T K)\boldsymbol{s} \qquad 1 \geq \lambda_i(K^T K) \geq 0$$

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For any vectors $\boldsymbol{h}, \boldsymbol{s}$ *let* K *be any* $n \times n$ *Hermitian matrix with spectrum in* $[0, 1]$*. Then*

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If now the filters and m_j *are fixed, for* $\boldsymbol{IMF}_j$ *the modes generated by* $\boldsymbol{s}$ *and for* $\boldsymbol{IMF}_j^*$ *generated by* $\boldsymbol{s} + \boldsymbol{h}$*, we have*

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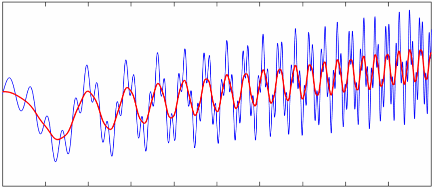
$\sum_j \|\boldsymbol{IMF}_j\|^2 \leq \|\boldsymbol{s}\|^2$. *Thus, there is a finite number of relevant IMF, i.e.* $\|\boldsymbol{IMF}_j\| > \eta$

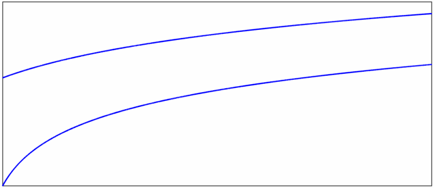
Resampled Iterative Filtering

Resampling

Signal $s(x)$



Instantaneous Frequencies



⏩ skip

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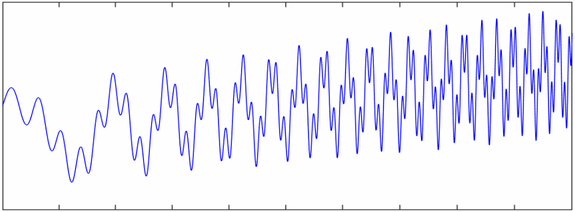

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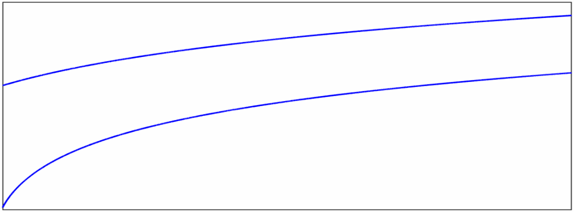

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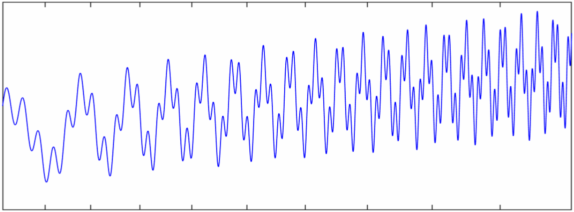

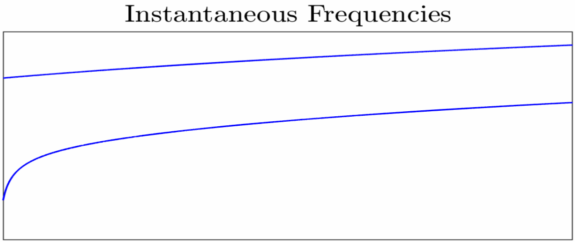

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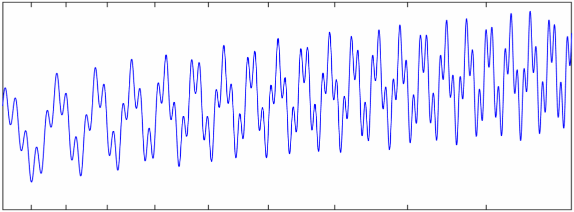

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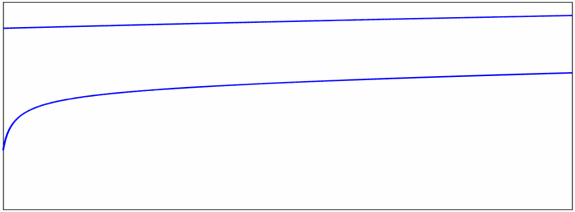

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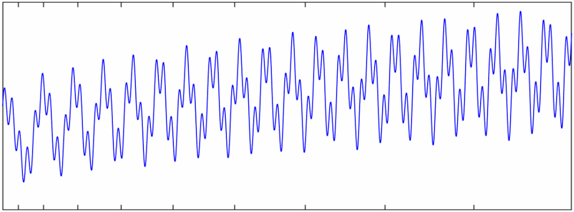

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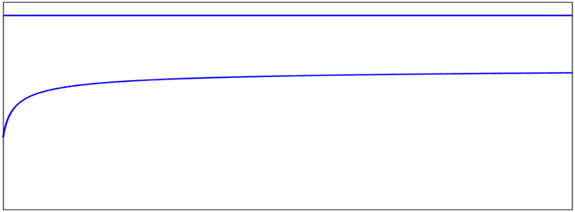

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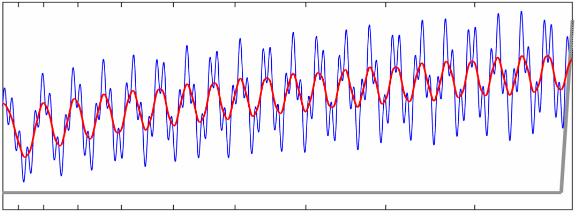

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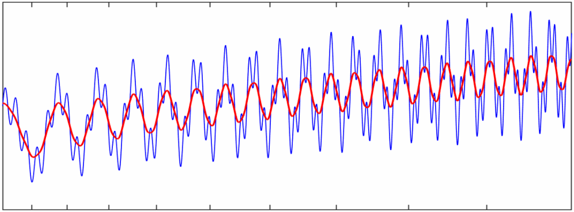

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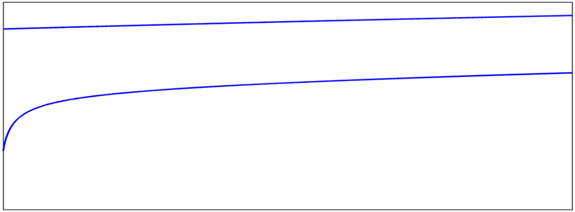

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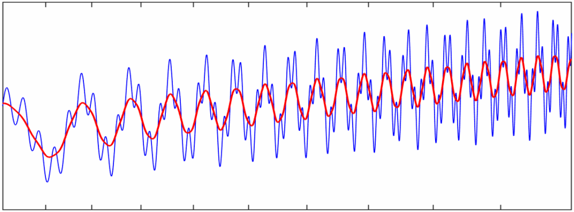

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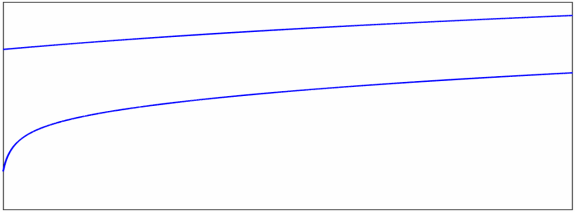

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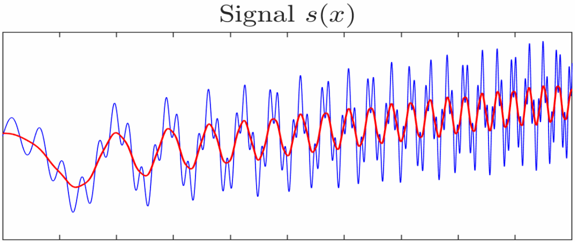

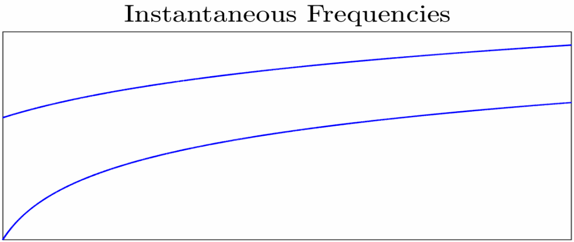

Resampling Function $G(y)$

Recall that in ALIF the length $\ell(x)$ is computed as $\xi_0/f(x)$ where $f(x)$ is the highest instantaneous frequency for the IMFs of the signal $s(x)$.
From now on $\xi_0 = 1$.

Example: The Instantaneous Frequency of $s(x) = \cos(\alpha(x))$ is $\alpha'(x)$ if it is regular enough. In this case, $\ell(x) = 1/\alpha'(x)$.

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In the Resampled IF (RIF), we instead operate a IF loop to the resampled stationary signal $s(G(y))$ where

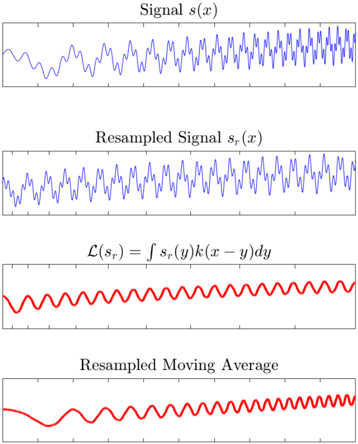
$$G^{-1}(z) = \int_0^z \frac{1}{\ell(x)} dx$$

Example: In the previous example, $G^{-1}(z) = \int_0^z \alpha'(x) = \alpha(z) - \alpha(0)$ so that

$$s(G(y)) = \cos(\alpha(G(y))) = \cos(\alpha(0) + y)$$

is a stationary signal with frequency equal to $\xi_0 = 1$

Resampled Iterative Filtering



Given the signal $s(x)$, compute the resampling

$$s_r(x) := s(G(x)) \qquad G^{-1}(z) = \int_0^z \frac{1}{\ell(x)}dx$$

and apply iteratively the filter through convolution

$$\mathcal{S}(f) := f(x) - \int f(y)k(x-y)dy$$

$$IMF = IMF \cup \{\mathcal{S}^{\infty}(s_r)(G^{-1}(x))\}$$

$$s = s - \mathcal{S}^{\infty}(s_r)(G^{-1}(x))$$

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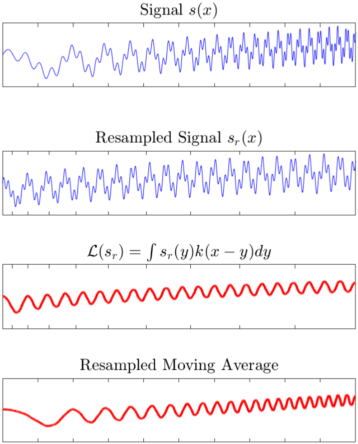

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We have an algorithm that is

- As flexible as ALIF and SALIF
- Efficient as Fast IF, the resampling is outside the iterations and has the same complexity as the FFT, thus way faster than ALIF and SALIF
- Differently from ALIF, $\mathcal{S}^\infty(s_r)$ is **always** convergent because it is an IF iteration. In particular, given a stopping criterion with $\delta > 0$ we have the same results that limit the number of iterations.

Theorem

Given $0 \le \widehat{k} \le 1,\ \delta > 0,\ s_r(x) \in L^2(\mathbb{R}),$ *then*

$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|s_r\|}$$

implies $\|\mathcal{S}^{m+1}(s_r) - \mathcal{S}^m(s_r)\| < \delta$

Theorem

For any $h, s_r \in L^2$

$$\|\mathcal{S}^m(s_r + h) - \mathcal{S}^m(s_r)\| \le \|h\|$$

Fast Discrete RIF

$$\widehat{\mathcal{S}^m(\boldsymbol{s}_r)} = \boldsymbol{k}^{\circ m} \circ \widehat{\boldsymbol{s}}_r$$

$$\|\mathcal{S}^{m+1}(\boldsymbol{s}_r) - \mathcal{S}^m(\boldsymbol{s})_r\| < \delta \iff \|\boldsymbol{k}^{\circ m} \circ (\boldsymbol{k} - \boldsymbol{e}) \circ \widehat{\boldsymbol{s}}_r\| < \delta$$

The stopping condition is checked on $\boldsymbol{k}$ and $\widehat{\boldsymbol{s}}_r$ with linear cost + 2 DFT

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We don't know if we can still recover

- Global perturbation results
- Intrinsic relation with $\widehat{\boldsymbol{s}}$
- Limited number of meaningful IMFs

Non-Stationary Error Bounds

Let us suppose that the signal $s(x)$ is a linear combination of non-stationary components

$$s(x) := \sum_{j=1}^{M} a_j g_j(x) \qquad g_j(x) = \cos(\alpha_j(x))$$

with $\alpha'_1(x) > \alpha'_2(x) > \cdots > \alpha'_M(x) > \epsilon > 0$ and $|a_j| \leq P$ for any j, and resampling

$$s_r(z) := \sum_{j=1}^{M} a_j h_j(z) \qquad h_j(z) = \cos(\alpha_j(\alpha_1^{-1}(2\pi s z)) = \cos(\beta_j(z))$$

where $h_j(x)$ are all 1-periodic and $h_1(z) = \cos(2\pi s z)$

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The IF Algorithm extracts as the first IMF the component $h_1(z)$ plus the coefficients of the components $h_j(z)$, $j \geq 2$ with frequency greater or equal than $\xi_0 = 2\pi s$. When the components are non-stationary, $\widehat{h_j}(z)$ for $j \geq 2$ may be non-zero also for high frequencies, thus we need an estimation of the error.

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$$\|f(x)_N\|_2^2 \leq \min\left\{ \left(\frac{b}{G + b + 2\pi} \right)^2, \frac{R^2}{\pi^3 G} \right\}$$

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$$\frac{b}{G + b + 2\pi} = \frac{\max_z \beta_j'(z)}{2\pi s} = \max_x \frac{\alpha_j'(x)}{\alpha_1'(x)} \qquad \text{(low for far frequencies)}$$

$$R = \max_z \beta_j'(z) - \min_z \beta_j'(z) = 2\pi s \left(\max_x \frac{\alpha_j'(x)}{\alpha_1'(x)} - \min_x \frac{\alpha_j'(x)}{\alpha_1'(x)} \right) \qquad \text{(zero if same shape)}$$

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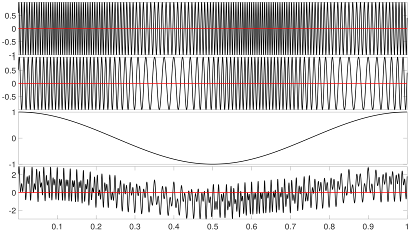
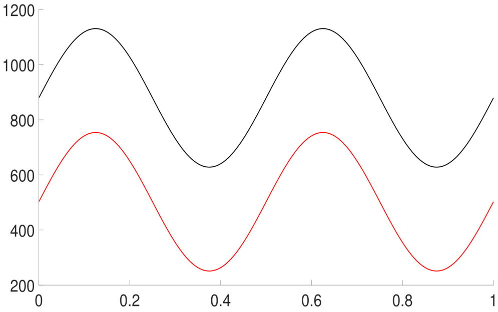
The method actually extracts only selected frequencies near ξ_0, with way less error

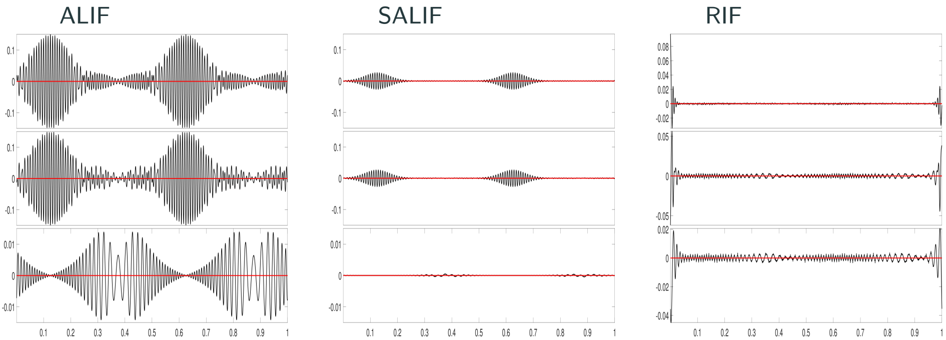
Numerical Experiments

Experiment 1

$N = 8000$
$h_1(x) = \cos(20\cos(4\pi t) - 160\pi t)$
$h_2(x) = \cos(20\cos(4\pi t) - 280\pi t)$
$h_3(x) = \cos(2\pi t)$
$h(x) = h_1(x) + h_2(x) + h_3(x)$

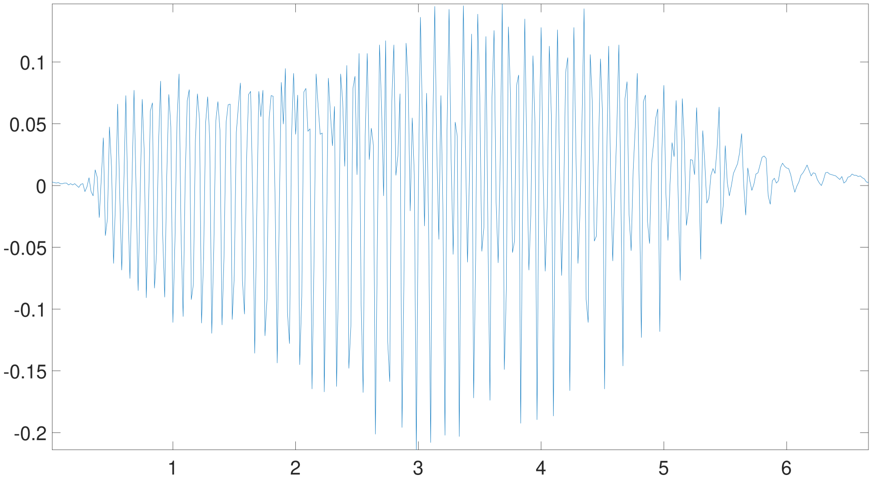




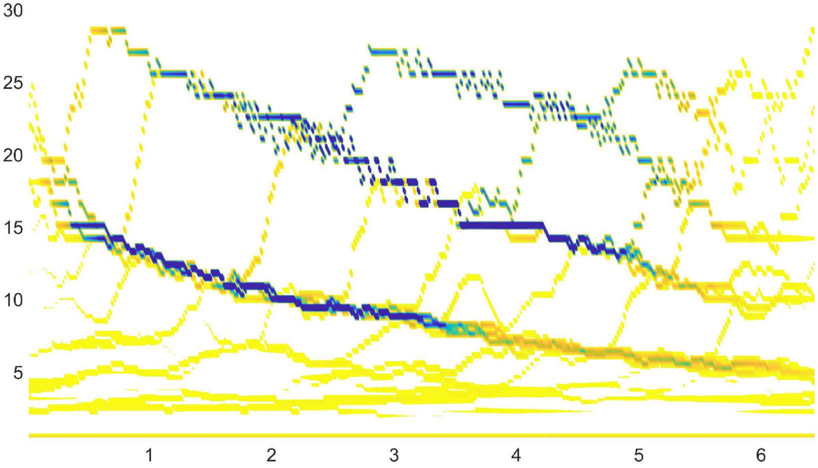


	Time	err1	err2	err3	Niter1	Niter2
ALIF	4.0860	0.070388	0.071158	0.008549	18	2
SALIF	19.7919	0.010054	0.010055	0.000161	353	5
RIF	1.4724	0.003426	0.003292	0.000908	81	11

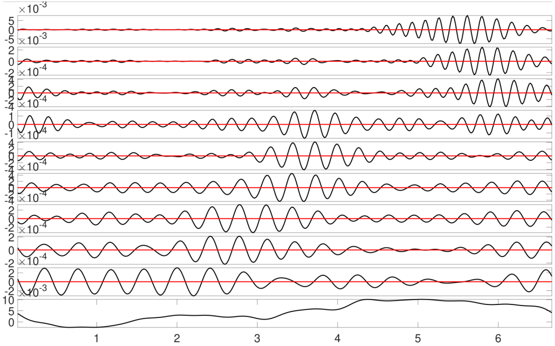
Experiment 2



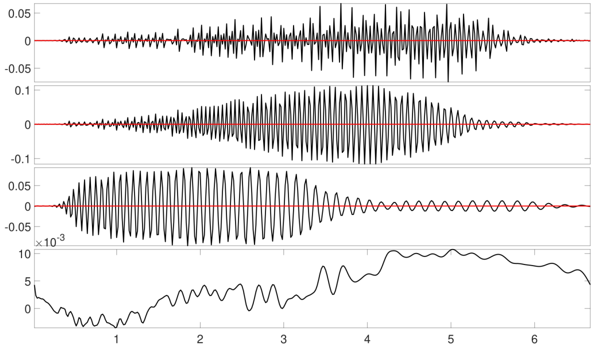
Experiment 2



IF



RIF



Conclusions and Future Works

We developed Algorithms and Theory for

- SALIF - Stable, Flexible, Convergent but very Slow
- RIF - Flexible, Convergent, Fast but may introduce inaccuracies

Moreover RIF proves himself also Aliasing-Free and we also expanded the theory of IF.

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Still to do:

- Better exploit the order of zero of the filter
- Further analysis of IF for non-stationary and AM components
- We can use RIF to better study ALIF through the relation between $G(x)$ and $\ell(x)$
- Better ways to compute $G(x)$ without relying on $\ell(x)$
- Improve the error bounds, since they prove to be empirically better
- How perturbation affect the output of RIF

Thank You!

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