The limit empirical spectral distribution of complex matrix polynomials

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Introduction on Spectral Distributions

Empirical Spectral Distribution (ESD)

Let X be a random variable with mean 0 and variance 1. A_n is a $n \times n$ random matrix whose entries are normalized i.i.d. copies of X:

$$A_n = \frac{1}{\sqrt{n}} \begin{bmatrix} X_{11} & \dots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \dots & X_{nn} \end{bmatrix}$$

Its empirical spectral distribution is a random variable taking values on the space of probability measures on $\mathbb{C}$. Specifically,

$$\mu_{A_n}(\omega) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(\omega)}$$

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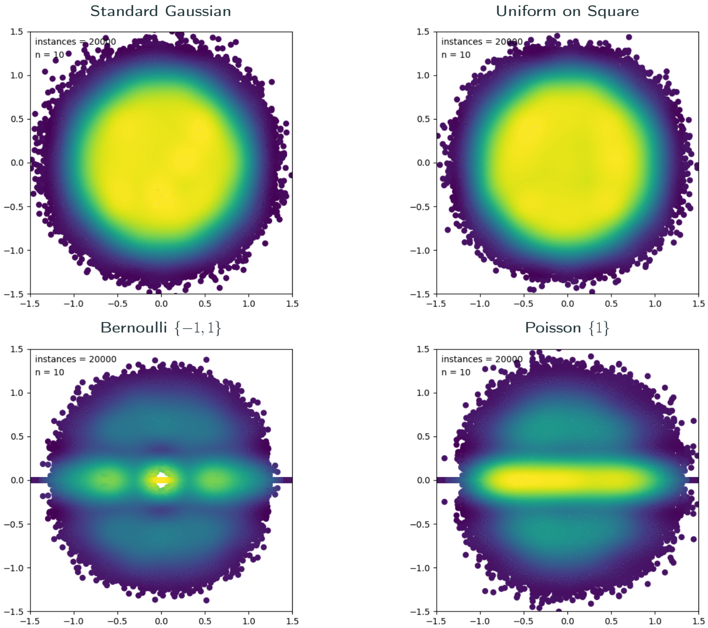

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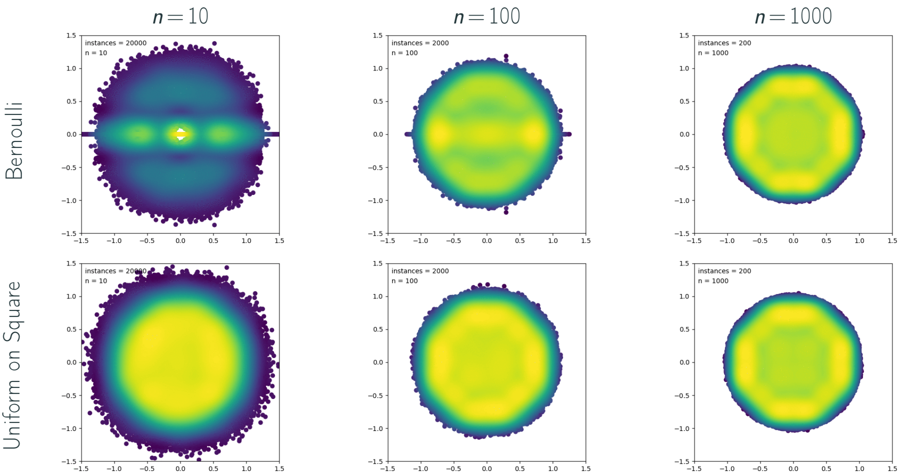

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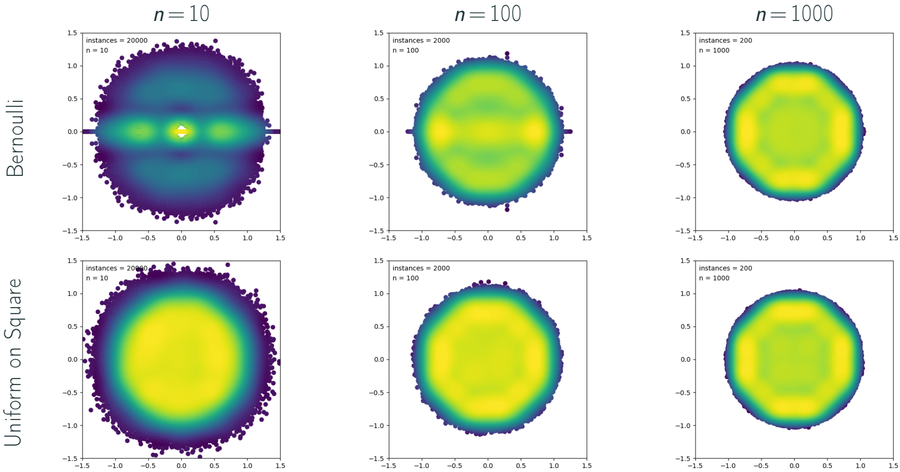

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Limit ESD and Circle Law

We say that μ is a limit ESD if $\mu_{A_n} \to \mu$ weakly as $n \to \infty$, i.e., if for all real valued continuous functions f with compact support

$$\mathbb{E}_{\mu_{A_n}}[f] \to \mathbb{E}_\mu[f] \quad \text{for } n \to \infty.$$

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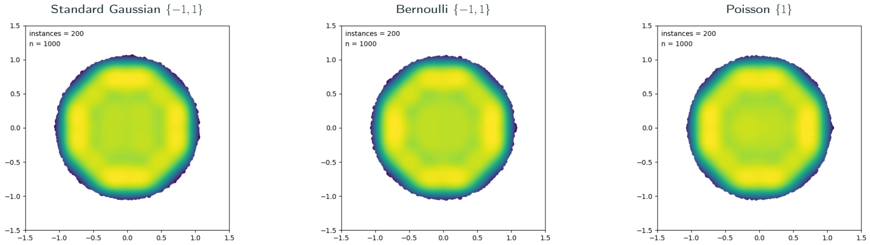
Circle Law [Tao and Vu, 2010]

Let A_n be an $n \times n$ random matrix with i.i.d. complex random entries of mean 0 and variance $1/n$. Then, almost surely, the limit ESD of A_n for $n \to \infty$ is the uniform distribution on the unit disk.



Limit ESD and Polynomial Roots

Let $p_k(z)$ be the degree k random polynomial whose coefficients are i.i.d. copies of X random variable with mean 0 and variance 1:

$$p_k(z) = X_0 + X_1 z + X_2 z^2 + \cdots + X_k z^k$$

Its roots are eigenvalues of the companion matrix.

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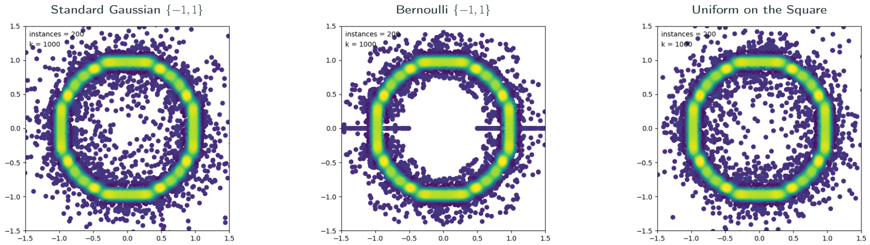
Theorem [Edelman, Kac, Kostlan, Shub, Smale...]

Almost surely, the limit ESD of the roots of $p_k(z)$ for $k \to \infty$ is the uniform distribution on the unit circle.



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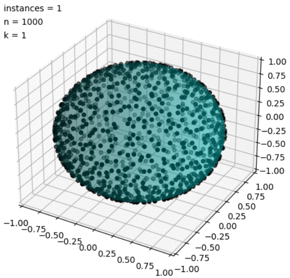
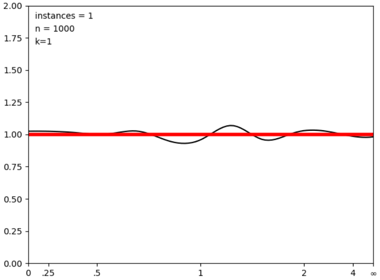
Theorem [Edelman, Kostlan, Krishnapur...]

Let $A + Bz$ be an $n \times n$ random pencil where the entries of A and B are all i.i.d. standard Gaussian complex random variables. Then, the empirical spectral distribution of the generalized eigenvalues of the pencil is (after a stereographic projection) the uniform distribution on the Riemann sphere.





Matrix Polynomials

Random Matrix Polynomials

We now consider a random matrix polynomial:

$$P_{n,k}(z) = \sum_{i=0}^{k} C_i z^i$$

where the coefficients C_i are random $n \times n$ matrices.

The Finite Eigenvalues of $P_{n,k}(z)$ are the roots of the polynomial $\det(P_{n,k}(z))$. If $\det(C_k) \neq 0$, then there are nk finite eigenvalues, corresponding with the spectrum of its random companion matrix

$$\begin{bmatrix} -C_k^{-1}C_{k-1} & \dots & -C_k^{-1}C_2 & -C_k^{-1}C_1 & -C_k^{-1}C_0 \\ I & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & 0 \end{bmatrix}$$

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Theorem [Noferini and GB, 2021]

Consider a random vector $[X_0, X_1, \ldots, X_k]$ where all X_j are independent random variables with zero mean and unit variance, not necessarily with the same distribution. Let $\alpha_0, \ldots, \alpha_k$ be complex constants with $\alpha_k \neq 0$. Suppose C_j are $n \times n$ random matrices whose entries are independent copies of X_j, and construct the random matrix polynomial

$$P_n(z) = \sum_{j=0}^{k} \alpha_j C_j z^j.$$

Then, for $n \to \infty$, the ESD of $P_n(z)$ converges almost surely to a probability measure μ with density

$$f(z) = \frac{1}{4k\pi} \Delta_z \ln \left(\sum_{i=0}^{k} |\alpha_i|^2 |z|^{2i} \right).$$

Corollaries

- **Kac Polynomials**. If $|\alpha_j| = 1$, then

$$f(z) = \frac{1}{\pi k}\left(\frac{1}{(|z|^2-1)^2} - \frac{(k+1)^2|z|^{2k}}{(|z|^{2k+2}-1)^2}\right).$$

When $k = 1$, then it is the uniform measure on the RS for any pencil (not just Gaussian).

- **Binomial or Elliptic Polynomials**. If $|\alpha_j|^2 = \binom{k}{j}$, then

$$f(z) = \frac{1}{\pi(|z|^2+1)^2} \text{ (Uniform on RS).}$$

- **Flat or Weyl Polynomials**. If $|\alpha_j|^2 = k^j (j!)^{-1}$, then

$$f(z) = \frac{1-(-k|z|^2+k+1)k|z|^{2k}\Gamma^{-1} - k|z|^{4k+2}\Gamma^{-2}}{\pi},$$

where

$$\Gamma = e^{k|z|^2}\int_{|z|^2}^{\infty}(e^{-s}s)^k \,\mathrm{d}s.$$

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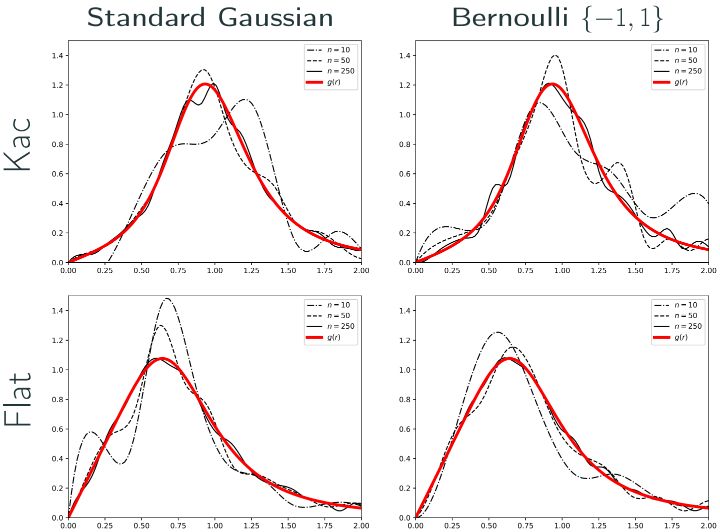
Experiments $k = 5$



Theorem [Noferini and GB, 2021]

Suppose $X_0^{(k)}, X_1^{(k)}, \dots, X_k^{(k)}$ are independent random variables with zero mean, unit variance, and uniformly bounded continuous distributions. Let also $\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_k^{(k)}$ be sequences of complex numbers with $\alpha_k^{(k)} = 1$. Let

$$P_{n,k}(z) = \sum_{j=0}^{k} \alpha_j^{(k)} C_j^{(k)} z^j,$$

where, for $j = 0, \dots, k$ every coefficient $C_j^{(k)}$ is an $n \times n$ random matrix whose entries are i.i.d. copies of $X_j^{(k)}$. If $n = O(k^P)$ for some $P > 0$, then the ESDs of $P_{n,k}$ converge almost surely as $k \to \infty$ to a probability measure μ with density

$$f(z) = \frac{1}{4\pi} \Delta_z \left[\lim_{k\to\infty} \frac{1}{k} \ln \left(\sum_{j=0}^{k} |\alpha_j^{(k)}|^2 |z|^{2i} \right) \right]$$

whenever it exists.

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- **Hyperbolic Polynomials.** If $|\alpha_j^{(k)}|^2 = \Gamma(d+j)/(\Gamma(d)j!)$ where $d > 0$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t$, then μ is the uniform measure on the unit circle.

Conjecture: It holds also if $n \notin O(k^P)$.

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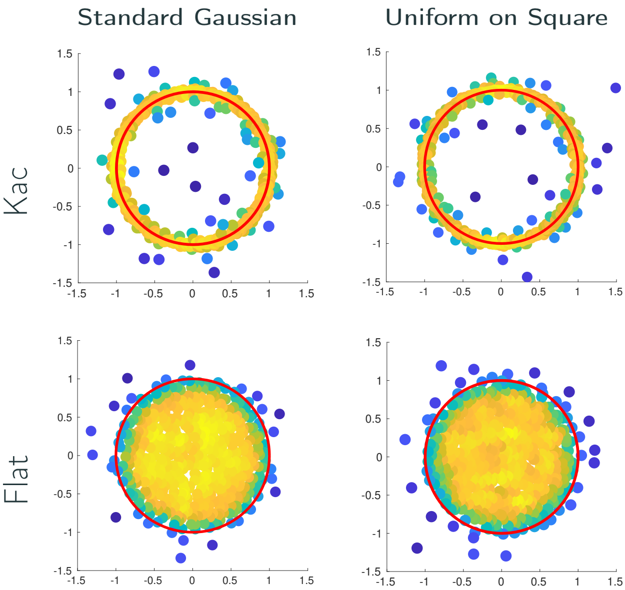
Experiments $n = 5,\ k = 100$



Idea of Proof

Let $s_k(z)^2$ be the variance of any entry of $P_{n,k}(z)$.

$$\frac{1}{nk}\ln|\det(P_{n,k}(z))| = \begin{cases} \frac{1}{k}\ln(s_k(z)) + \frac{1}{nk}\sum_{i=1}^{n}\ln(\sigma_i(P_{n,k}(z)/s_k(z))) \\ \frac{1}{nk}\ln|\det(C_k)| + \frac{1}{nk}\sum_{i=1}^{nk}\ln|z-\lambda_i(P_{n,k})| \end{cases}$$

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Both for $n\to\infty$ and $k\to\infty$ we have

$$\frac{1}{k}\left|\int_{\mathbb{R}}\ln(x)\,\mathrm{d}\nu_{P_{n,k}(z)/s_k(z)} - \int_{\mathbb{R}}\ln(x)\,\mathrm{d}\nu_{C_k}\right| \to 0$$

and from the theory of Logarithmic Potentials

$$2\pi\lim_{k|n\to\infty}\mu_{P_{n,k}} = \Delta_z\lim_{k|n\to\infty}\int_{\mathbb{C}}\ln|z-z'|\,\mathrm{d}\mu_{P_{n,k}}(z') = \Delta_z\lim_{k|n\to\infty}\frac{1}{k}\ln(s_k(z))$$

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If there's time...

$$P_n(z) = Iz^k + \sum_{i=0}^{k-1} C_i z^i$$

where C_i are $n \times n$ random matrices whose entries are Gaussian random variables with mean 0 and variance 1.

Theorem [Noferini and GB, 2020]

Almost surely, as $n \to \infty$ the empirical spectral distribution of $P_n(\sqrt{n}z)$ converges to

$$\frac{k-1}{k}\delta_0 + \frac{1}{k}1_D$$

where $\delta_0, 1_D$ are the atomic distribution on zero and the uniform distribution on the unit disk respectively.

For $k \to \infty$, all the non-monic case results still hold.

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Thank You!

- Giovanni Barbarino and Vanni Noferini. **The limit empirical spectral distribution of complex matrix polynomials.** *Random Matrices: Theory and Applications*, 09 2021.
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