Higher order spectral symbols and eigenvalues approximation for Toeplitz matrices

March 11, 2018

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Spectral Symbol

Given
$$A_n \in \mathbb{C}^{n \times n}$$
 and $f : [0, 1] \to \mathbb{C}$ measurable,

 $\{A_n\}_n \sim_\lambda f(x)$

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$$\Lambda(A_n) = \{\lambda_{\sigma_n(1),n}, \dots, \lambda_{\sigma_n(n),n}\}$$

$$\lambda_{1,n} \le \lambda_{2,n} \le \dots \le \lambda_{n,n}$$

$$\lambda_{\sigma_n(i),n} \sim f\left(\frac{i}{n+1}\right) \quad \forall i$$

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Quantile

Given $f:[0,1] \to \mathbb{R}$ its quantile $k:[0,1] \to \mathbb{R}$ is increasing and

$$\lambda \{ x : f(x) \le M \} = \lambda \{ x : k(x) \le M \} \quad \forall M$$

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 $\lambda_{1,n} \leq \lambda_{2,n} \leq \cdots \leq \lambda_{n,n}$

If k is continuous

$$\lambda_{i,n} \sim k\left(rac{i}{n+1}
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Szegő Theorem

Let $f \in L^1[-\pi, \pi]$ real-valued function $T_n(f) = [f_{i-j}]_{i,j} \implies \{T_n(f)\}_n \sim_{\lambda} f$ $\Lambda(T_n(f)) \subseteq [\inf \operatorname{ess} f, \sup \operatorname{ess} f]$

If f is bounded and its quantile k on $[0, \pi]$ is continuous,

$$\lambda_{i,n} - k\left(rac{i\pi}{n+1}
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Hope

$$\lambda_{i,n} = k\left(\frac{i\pi}{n+1}\right) + \frac{c_1\left(\frac{i\pi}{n+1}\right)}{n+1} + \frac{c_2\left(\frac{i\pi}{n+1}\right)}{(n+1)^2} + \dots$$

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Given f continuous, let k be the quantile on $[0,\pi]$ and compute

$$(n+1)\left[\lambda_{i,n}(T_n(f))-k\left(\frac{i\pi}{n+1}\right)\right]$$

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Main Theorem

Suppose the function $f : [0, 2\pi] \to \mathbb{R}$ is C_{per}^m and



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If $\alpha = m - 3$ and k is the quantile of f on $[0, \pi]$, then

$$\lambda_{i,n} = k\left(\frac{i\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{i\pi}{n+1}\right)}{(n+1)^s} + o\left(\frac{1}{n^{\alpha}}\right)$$

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Toeplitz Operator on $L^2_{(n)}$

Aims

Suppose
$$f(x) = f(2\pi - x)$$
 ($\implies f|_{[0,\pi]} \equiv k$ quantile)

• Find a characterization of $\Lambda(T_n(f))$

• Find an expansion of $f^{-1}(\lambda_{i,n}) - \frac{i\pi}{n+1}$

$$f(\theta) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta} \equiv \sum_{k=-\infty}^{\infty} f_k t^k = f(t) \qquad P_n\left(\sum_{k=-\infty}^{\infty} a_k t^k\right) = \sum_{k=0}^{n-1} a_k t^k$$
$$L_{(n)}^2 := P_n(L^2)$$

Toeplitz

Given $f \in L^2$, then $T_n(f): L^2_{(n)} o L^2_{(n)}$ is a linear operator $T_n(f)g := P_n(fg)$

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$$\lambda \in \Lambda(T_n(f)) \iff \exists g \in L^2_{(n)} : T_n(f(t) - \lambda)g(t) = 0$$



$$h(t) = P_{n+2}\left[b(t,\widetilde{\theta})\frac{(t-e^{i\widetilde{\theta}})(e^{-i\widetilde{\theta}}-t)}{t}tg(t)\right] = T_{n+2}(b(\cdot,\widetilde{\theta}))\widetilde{g}(t)$$

$$h(t) = \begin{pmatrix} * & * & * \\ * & T_n(f - \lambda) & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix} = \begin{pmatrix} h_0 \\ 0 \\ h_{n+1} \end{pmatrix} = h_0 + h_{n+1}t^{n+1}$$

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$$h(t) := T_{n+2}(f(\cdot) - f(\widetilde{\theta}))(tg(t)) \neq 0$$



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$$\lambda \in \Lambda(T_n(f)) \implies T_{n+2}(b(\cdot,\widetilde{\theta}))\widetilde{g}(t) = h_0 + h_{n+1}t^{n+1}$$
$$\implies (t - e^{i\widetilde{\theta}})(e^{-i\widetilde{\theta}} - t)g(t) = T_{n+2}(b(\cdot,\widetilde{\theta}))^{-1}(h_0 + h_{n+1}t^{n+1})$$

 $b(t, \theta)$ is real, so $T_{n+2}(b(\cdot, \theta))^{-1}$ is hermitian and symmetric wrt the antidiagonal. If $\Phi_{n+2}(t, \tilde{\theta}) := T_{n+2}(b(\cdot, \tilde{\theta}))^{-1}(1)$, then

$$(t-e^{i\widetilde{\theta}})(e^{-i\widetilde{\theta}}-t)g(t)=h_0\Phi_{n+2}(t,\widetilde{\theta})+h_{n+1}t^{n+1}\overline{\Phi_{n+2}(t,\widetilde{\theta})}$$

$$\implies \begin{pmatrix} \Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta}) & e^{(n+1)i\widetilde{\theta}}\overline{\Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta})} \\ \Phi_{n+2}(e^{-i\widetilde{\theta}},\widetilde{\theta}) & e^{-(n+1)i\widetilde{\theta}}\overline{\Phi_{n+2}(e^{-i\widetilde{\theta}},\widetilde{\theta})} \end{pmatrix} \begin{pmatrix} h_0 \\ h_{n+1} \end{pmatrix} = 0$$

$$\frac{\Phi_{n+2}(e^{-i\widetilde{\theta}},\widetilde{\theta})\Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta})}{\Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta})} = e^{2(n+1)i\widetilde{\theta}}$$

$$\lambda \in \Lambda(T_n(f)) \implies T_{n+2}(b(\cdot, \tilde{\theta}))\tilde{g}(t) = h_0 + h_{n+1}t^{n+1}$$
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 $b(t, \widetilde{\theta})$ is real, so $T_{n+2}(b(\cdot, \widetilde{\theta}))^{-1}$ is hermitian and symmetric wrt the antidiagonal. If $\Phi_{n+2}(t, \widetilde{\theta}) := T_{n+2}(b(\cdot, \widetilde{\theta}))^{-1}(1)$, then

$$(t-e^{i\widetilde{ heta}})(e^{-i\widetilde{ heta}}-t)g(t)=h_0\Phi_{n+2}(t,\widetilde{ heta})+h_{n+1}t^{n+1}\overline{\Phi_{n+2}(t,\widetilde{ heta})}$$

$$\implies \begin{pmatrix} \Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta}) & e^{(n+1)i\widetilde{\theta}}\overline{\Phi_{n+2}(e^{i\widetilde{\theta}},\widetilde{\theta})} \\ \Phi_{n+2}(e^{-i\widetilde{\theta}},\widetilde{\theta}) & e^{-(n+1)i\widetilde{\theta}}\overline{\Phi_{n+2}(e^{-i\widetilde{\theta}},\widetilde{\theta})} \end{pmatrix} \begin{pmatrix} h_0 \\ h_{n+1} \end{pmatrix} = 0$$

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nan

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$$\implies \begin{pmatrix} \Phi_{n+2}(e^{i\widetilde{\theta}}, \widetilde{\theta}) & e^{(n+1)i\widetilde{\theta}} \overline{\Phi_{n+2}(e^{-i\widetilde{\theta}}, \widetilde{\theta})} \\ \Phi_{n+2}(e^{-i\widetilde{\theta}}, \widetilde{\theta}) & e^{-(n+1)i\widetilde{\theta}} \overline{\Phi_{n+2}(e^{-i\widetilde{\theta}}, \widetilde{\theta})} \end{pmatrix} \begin{pmatrix} h_0 \\ h_{n+1} \end{pmatrix} = 0$$

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Toeplitz Operator on L^2_+

$$T(f) = [f_{i-j}]_{i=1,...,\infty}^{j=1,...,\infty} \qquad P\left(\sum_{k=-\infty}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} a_k t^k$$

Toeplitz

Given $f \in L^2$, then $T_n(f): L^2_+ \to L^2_+$ is a linear operator T(f)g := P(fg)

Wiener Hopf

Suppose $a(t) \in L^2$ where $a(t) \neq 0$ and wind(a, 0) = 0. Then $a = a_+a_ a_+ \in L^2_+$, $a_- \in L^2_ T(a)^{-1} = T(a^{-1}_+)T(a^{-1}_-)$

 $T(a)^{-1}1 = T(a_{+}^{-1})T(a_{-}^{-1})1 = P(a_{+}^{-1}P(a_{-}^{-1})) = a_{+}^{2}(\mathbf{x})a_{+}^{-1}(t) = \mathbf{x}$

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$$a = a_{+}a_{-}$$
 $a_{+} \in L^{2}_{+}, a_{-} \in L^{2}_{-}$
 $T(a)^{-1} = T(a^{-1}_{+})T(a^{-1}_{-})$

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$$\Phi_{n+2}(t,\widetilde{\theta}) = T_{n+2}(b(\cdot,\widetilde{\theta}))^{-1}(1) = b(t,\widetilde{\theta})^{-1}_{+} + o(n^{4-m})$$

$\exp(2i\eta(\widetilde{\theta}) + 2iR^{(n)}(\overline{\theta})) = \exp(2(n+1)i\widetilde{\theta})$

heta Approximation

 λ is an eigenvalue of $T_n(f)$ iff there exists $j \in \mathbb{Z}$ such that $\widetilde{\theta} = f^{-1}(\lambda) \in (0, \pi)$ and satisfies

$$(n+1)\widetilde{\theta} - \eta(\widetilde{\theta}) - R^{(n)}(\widetilde{\theta}) = j\pi$$

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$\tilde{\theta}$ Approximation

 λ is an eigenvalue of $T_n(f)$ iff there exists $j \in \mathbb{Z}$ such that $\widetilde{\theta} = f^{-1}(\lambda) \in (0, \pi)$ and satisfies

$$(n+1)\widetilde{\theta} - \eta(\widetilde{\theta}) - \mathcal{R}^{(n)}(\widetilde{\theta}) = j\pi$$

•
$$\eta(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\log b(\tau, x)}{\tau - e^{ix}} d\tau - \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\log b(\tau, x)}{\tau - e^{-ix}} d\tau, \ \eta(0) = \eta(\pi) = 0$$

• $R^{(n)}(x) = o(n^{4-m}), \quad R^{(n)}(0) = R^{(n)}(\pi) = 0$

$$G(\widetilde{\theta}) := (n+1)\widetilde{\theta} - \eta(\widetilde{\theta}) - R^{(n)}(\widetilde{\theta}) = j\pi$$

$G(0) = 0, \quad G(\pi) = (n+1)\pi \implies \forall j \quad \exists ! \theta_{j,n} : G(\theta_{j,n}) = j\pi$

$$\theta_{j,n} = \frac{j\pi}{n+1} + \frac{\eta(\theta_{j,n})}{n+1} + o(n^{3-m}) \qquad f(\theta_{j,n}) = \lambda_{j,n}$$

$$\begin{aligned} \theta_{j,n} &= \frac{j\pi}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)}{n+1} + o(n^{-1}) \\ &= \frac{j\pi}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)}{n+1} + \frac{\eta\left(\frac{j\pi}{n+1}\right)\eta'\left(\frac{j\pi}{n+1}\right)}{(n+1)^2} + o(n^{-2}) \end{aligned}$$

 $=\frac{j\pi}{n+1}+\sum_{s=1}^{m-3}\frac{d_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s}+o\left(\frac{1}{n^{m-3}}\right)$

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 $= \ldots$

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$$= f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{m-3} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + o\left(\frac{1}{n^{m-3}}\right)$$

Conjecture

RCTP

 $u(\theta)$ is a Real Cosine Trigonometrical Polynomial if

$$u(\theta) = u_0 + 2\sum_{i=1}^m u_i \cos(k\theta)$$

Assumption

Let f = u/v increasing on $[0, \pi]$, where u, v are RCTPs and $v \neq 0$

$$\lambda_j(T_n(f)) = f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + E_{j,n,\alpha}$$

• $c_k \in C^{\alpha-k+1}[0,\pi]$
• $E_{j,n,\alpha} = O(n^{-\alpha-1})$

Spoiler: False even if v = 1 for high $\alpha \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a$

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Notation: Given $j_1 \leq n_1$ positive integers,

•
$$n_k := 2^{k-1}(n_1+1) - 1$$
 $j_k := 2^{k-1}j_1$
 $h_k := (n_k+1)^{-1} = 2^{1-k}h_1$

•
$$d_{j,n} := j\pi/(n+1) \implies d_{j_k,n_k} = d_{j_1,n_1}$$

Interpolation

Solve for every $k = 1, \dots, \alpha$ and $j_1 = 1, \dots, n_1$ $\lambda_{j_k}(\mathcal{T}_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} \widetilde{c}_{s,j_1} h_k^s$ and interpolate for every $k = 1, \dots, \alpha$

 $(d_{1,n_1},\widetilde{c}_{s,1}),(d_{2,n_1},\widetilde{c}_{s,2}),\ldots,(d_{n_1,n_1},\widetilde{c}_{s,n_1})$

 $\widetilde{c}_{s,h} \sim c_{s,h}$, but \widetilde{c}_{s} is NOT a good approximation of $c_{s} \sim 10^{-3}$

$$\lambda_j(T_n(f)) = f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + E_{j,n,\alpha}$$

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Interpolation

Solve for every $k = 1, ..., \alpha$ and $j_1 = 1, ..., n_1$ $\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} \widetilde{c}_{s,j_1} h_k^s$ and interpolate for every $k = 1, ..., \alpha$ $(d_{1,n_1}, \widetilde{c}_{s,1}), (d_{2,n_1}, \widetilde{c}_{s,2}), ..., (d_{n_1,n_1}, \widetilde{c}_{s,n_1})$

 $\widetilde{c}_{e,L}\sim c_{e,L}$, but \widetilde{c}_{e} is NOT a good approximation at $c_{e,L}$ and $c_{e,L}$ but $\widetilde{c}_{e,L}$ is NOT a good approximation at $c_{e,L}$ and $c_{e,L}$

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} c_s(d_{j_1,n_1}) h_k^s + E_{j_k,n_k,\alpha}$$

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$$d_{j,n} := j\pi/(n+1) \implies d_{j_k,n_k} = d_{j_1,n_1}$$

Interpolation

Solve for every $k = 1, \ldots, \alpha$ and $j_1 = 1, \ldots, n_1$

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\infty} \widetilde{c}_{s,j_1} h_k^s$$

and interpolate for every $k = 1, \ldots, c$

$$(d_{1,n_1}, \widetilde{c}_{s,1}), (d_{2,n_1}, \widetilde{c}_{s,2}), \dots, (d_{n_1,n_1}, \widetilde{c}_{s,n_1})$$

 $\widetilde{c}_{s,j_1}\sim c_{s,j_1}$, but \widetilde{c}_s is NOT a good approximation of $c_{s_{\Xi}}$, $c_{s_{\Xi}}$

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} c_s(d_{j_1,n_1}) h_k^s + E_{j_k,n_k,\alpha}$$

Notation: Given $j_1 \leq n_1$ positive integers,

•
$$n_k := 2^{k-1}(n_1+1) - 1$$

 $h_k := (n_k+1)^{-1} = 2^{1-k}h_1$
 $j_k := 2^{k-1}j_1$

•
$$d_{j,n} := j\pi/(n+1) \implies d_{j_k,n_k} = d_{j_1,n_1}$$

Interpolation

Solve for every $k = 1, \ldots, \alpha$ and $j_1 = 1, \ldots, n_1$

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{r} \widetilde{c}_{s,j_1} h_k^s$$

and interpolate for every $k=1,\ldots, lpha$

$$(d_{1,n_1}, \widetilde{c}_{s,1}), (d_{2,n_1}, \widetilde{c}_{s,2}), \dots, (d_{n_1,n_1}, \widetilde{c}_{s,n_1})$$

 $\widetilde{c}_{s,j_1}\sim c_{s,j_1}$, but \widetilde{c}_s is NOT a good approximation of $c_{s_{\Xi}}$,

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} c_s(d_{j_1,n_1}) h_k^s + E_{j_k,n_k,\alpha}$$

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Solve for every $k = 1, \ldots, \alpha$ and $j_1 = 1, \ldots, n_1$

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 $\implies \sum_{s=1}^{\alpha} \left[c_s \left(d_{j_1, n_1} \right) - \widetilde{c}_{s, j_1} \right] h_k^{s-1} = -E_{j_k, n_k, \alpha} / h_k = O(h_k^{\alpha}) = O(h_1^{\alpha})$

 $\implies A \operatorname{diag}(1, h_1, \dots, h_1^{\alpha-1})(\boldsymbol{c} - \widetilde{\boldsymbol{c}}) = O(h_1^{\alpha})$

$$\implies (c - \widetilde{c})_k = O(h_1^{\alpha - k + 1})$$

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Better Approximation

$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} c_s(d_{j_1,n_1}) h_k^s + E_{j_k,n_k,\alpha}$$
$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} \widetilde{c}_{s,j_1} h_k^s$$

Focus on
$$\lambda_j(T_n(f))$$
. Let
 $\{d^{(1)}, d^{(2)}, \dots, d^{(\alpha-k+1)}\} \subseteq \{d_{1,n_1}, d_{2,n_1}, \dots, d_{n_1,n_1}\}$
the closest points to $d_{j,n}$, and interpolate

$$(d^{(1)},\widetilde{c}_k(d^{(1)})),(d^{(2)},\widetilde{c}_k(d^{(2)})),\ldots,(d^{(\alpha-k+1)},\widetilde{c}_k(d^{(\alpha-k+1)}))$$

Eigenvalue Approximation

Given $p_{k,j,n}$ the resulting polynomials,

$$\widetilde{\lambda}_j(T_n(f)) := f(d_{j,n}) + \sum_{r=1}^{lpha} p_{s,j,n}(d_{j,n})h$$

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$$\lambda_{j_k}(T_{n_k}(f)) = f(d_{j_1,n_1}) + \sum_{s=1}^{\alpha} c_s(d_{j_1,n_1}) h_k^s + E_{j_k,n_k,\alpha}$$
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Eigenvalue Approximation

Given $p_{k,j,n}$ the resulting polynomials,

$$\widetilde{\lambda}_j(\mathcal{T}_n(f)) := f(d_{j,n}) + \sum_{s=1}^{lpha} p_{s,j,n}(d_{j,n})h^s$$

$$|c_k(d_{j,n})-p_{k,j,n}(d_{j,n})|\leq B_lpha h_1^{lpha-k+1}$$

If $q_{k,j,n}$ interpolates

$$(d^{(1)}, c_k(d^{(1)})), (d^{(2)}, c_k(d^{(2)})), \dots, (d^{(\alpha-k+1)}, c_k(d^{(\alpha-k+1)}))$$

then

$$|c_k(d_{j,n}) - q_{k,j,n}(d_{j,n})| \le \|c_k\|_\infty (h_1\pi)^{lpha - k + 1} rac{(lpha - k + 1)^{lpha - k + 1}}{(lpha - k + 1)!}$$

$$\begin{aligned} |p_{k,j,n}(d_{j,n}) - q_{k,j,n}(d_{j,n})| &\leq \sum_{r=1}^{\alpha-k+1} \prod_{s\neq r} \frac{|d_{j,n} - d^{(s)}|}{|d^{(r)} - d^{(s)}|} |c_k(d^{(r)}) - \widetilde{c}_k(d^{(r)})| \\ &\leq A_\alpha h_1^{\alpha-k+1} (\alpha-k+1)^{\alpha-k+1} \end{aligned}$$

$$\implies |c_k(d_{j,n}) - p_{k,j,n}(d_{j,n})| \le B_\alpha h_1^{\alpha-k+1}$$

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$$\lambda_j(T_n(f)) = f(d_{j,n}) + \sum_{s=1}^{\alpha} c_s(d_{j,n}) h^s + E_{j,n,\alpha}$$
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 $|\lambda_j(T_n(f)) - \widetilde{\lambda}_j(T_n(f))| = O_\alpha(hh_1^\alpha)$

- The error decreases if $n \to \infty$
- The error decreases if $n_1
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- The error does not decrease if $\alpha
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It is better to keep α low also for the computational cost

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$$\begin{split} \lambda_j(T_n(f)) &= f(d_{j,n}) + \sum_{s=1}^{\alpha} c_s(d_{j,n}) h^s + E_{j,n,\alpha} \\ \widetilde{\lambda}_j(T_n(f)) &= f(d_{j,n}) + \sum_{s=1}^{\alpha} p_{s,j,n}(d_{j,n}) h^s \\ &|c_k(d_{j,n}) - p_{k,j,n}(d_{j,n})| \le B_{\alpha} h_1^{\alpha-k+1} \end{split}$$

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Input: $n > n_1 > \alpha$, $S \subseteq \{1, \ldots, n\}$, $f \in C^{\infty}_{ner}[-\pi, \pi]$ for $k = 1, \ldots, \alpha$ do Compute $eig(T_{n_{k}}(f))$ end for for $i_1 = 1, ..., n_1$ do Compute $\tilde{c}_{i_1} = V^{-1}[\lambda_{i_k}(T_{n_k}(f)) - f(d_{i_1,n_1})]_k$ end for for $i \in S$ do for $k = 1, \ldots, \alpha$ do Determine $\alpha - k + 1$ points $d^{(s)}$ closest to $d_{i,n}$. Compute $p_{k,i,n}(d_{i,n})$ where $p_{k,i,n}$ interpolates $(d^{(s)}, \tilde{c}_{\ell}(d^{(s)}))$ implicitly end for $\lambda_i(T_n(f)) = f(d_{i,n}) + \sum_{s=1}^{\alpha} p_{s,i,n}(d_{i,n}) h^s$ end for

$$\operatorname{Cost}: \sum_{k} O(\operatorname{eig}(T_{n_{k}}(f))) + O(\alpha^{2}n_{1}) + O(\alpha^{3}|S|)$$

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Parallel Cost : $O(eig(T_{n_{\alpha}}(f))) + O(\alpha^3)$

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Compute all $\widetilde{\lambda}_j(T_n(f))$ with n = 5000 and

$$f(\theta) = \frac{40 - 15\cos(\theta) - 24\cos(2\theta) - \cos(3\theta)}{1208 + 1191\cos(\theta) + 120\cos(2\theta) + \cos(3\theta)}$$

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Method	CPU time	max error
Algorithm with $n_1 = 50$, $\alpha = 4$	1.69	$\sim 10^{-7}$
Algorithm with $n_1=$ 50, $lpha=$ 4	2.77	$\sim 10^{-8}$
Algorithm with $n_1=$ 50, $lpha=$ 4	18.30	$\sim 10^{-9}$
Algorithm with $n_1 = 50$, $lpha = 4$	280.27	$\sim 10^{-11}$
MATLAB eig's function	1265.55	/

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Compute the first order symbol of

$$f(\theta) = 2 - \cos(\theta) - \cos(3\theta)$$

The previous results apply on the intervals of $[0, \pi]$ where f is **injective**.

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Counterexample

Assumption

Let f = u/v increasing on $[0, \pi]$, where u, v are RCTPs and $v \neq 0$

$$\lambda_j(T_n(f)) = f\left(\frac{j\pi}{n+1}\right) + \sum_{s=1}^{\alpha} \frac{c_s\left(\frac{j\pi}{n+1}\right)}{(n+1)^s} + E_{j,n,\alpha}$$

•
$$c_k \in C^{\alpha-k+1}[0,\pi]$$

•
$$E_{j,n,\alpha} = O(n^{-\alpha-1})$$

BBGM

 $f(\theta) = (\sin(\theta/2))^4$ (pentadiagonal) respects the hypothesis, but fails for $\alpha = 5$.

$$f''(0) = 0 \implies b(1,0) = 0 \implies \log(b), b_{\pm}^{-1} \text{singular}$$
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