Perturbations of Hermitian Matrices and Applications to Spectral Symbols

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Spectral Symbols

Simple Example

$$\begin{cases} u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\lambda_h(A_n) = 2 - 2\cos\left(\frac{h\pi}{n+1}\right)$$

→ The sequence $\{A_n\}_n$ has Spectral Symbol $k(t)$

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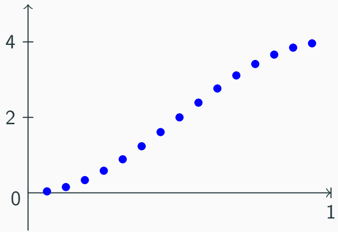

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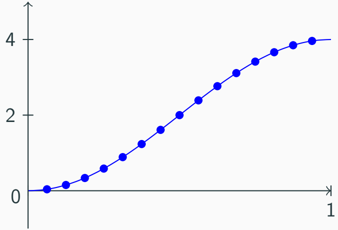

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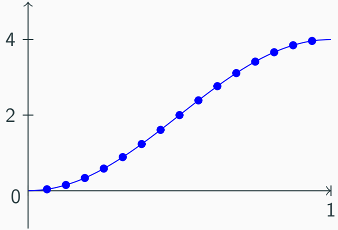

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→ **The sequence $\{A_n\}_n$ has Spectral Symbol $k(t)$**

Asymptotic Distribution

Spectral Symbol

Let $\{A_n\}_n$ a matrix sequence, and $k : D \subseteq \mathbb{R}^m \to \mathbb{C}$ measurable.

$$\{A_n\}_n \sim_\lambda k \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu(D)} \int_D F(k(t))\mathrm{dt}$$

for all $F \in C_c(\mathbb{C})$.

$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < k(t) < b\}}{\mu(D)}$$

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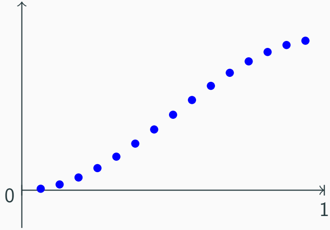

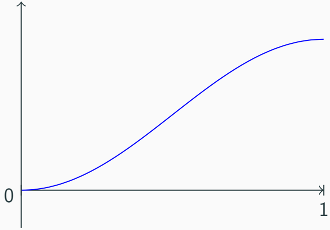

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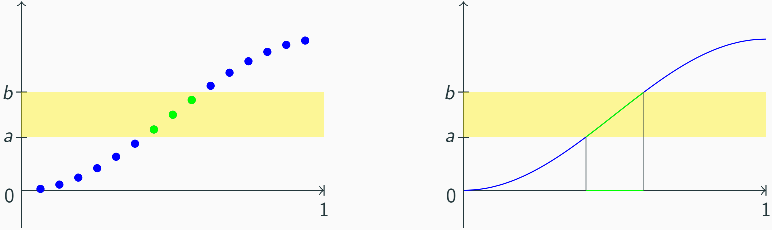

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Equivalent Definition

Let $\{A_n\}_n$ a matrix sequence and $k : [0,1] \to \mathbb{R}$ measurable.

$$\{A_n\}_n \sim_\lambda k \iff f(A_n) \xrightarrow{\mu} k$$

where $f(X)$ is the piecewise linear interpolator of $\Lambda(X)$ over $[0,1]$.

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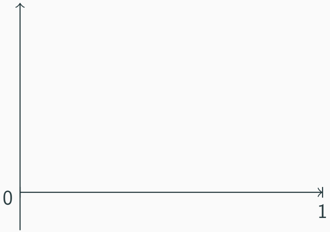

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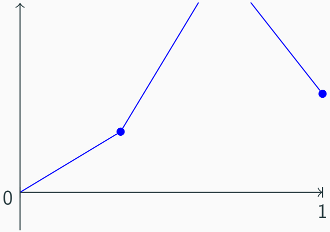

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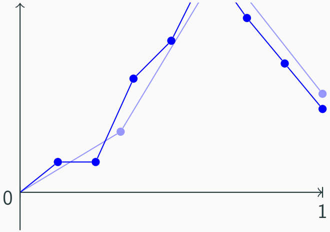

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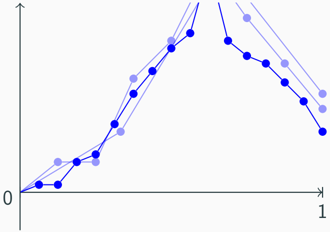

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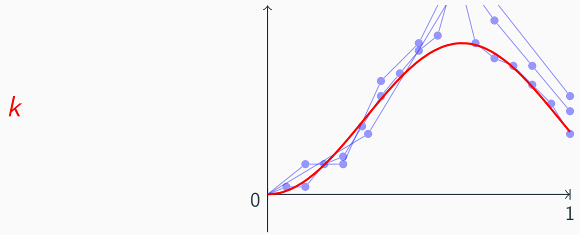

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Hermitian Perturbations

Zero Distributed Matrices

Let N_n and R_n be Hermitian matrices s.t.

- $\|N_n\| = o(1)$
- $\text{rk}(R_n) = o(n)$

Given A_n Hermitian matrices, we get

$$\{A_n\}_n \sim_\lambda k \iff \{A_n + N_n + R_n\}_n \sim_\lambda k$$

Idea of Proof

- Cauchy Interlacing Theorem

$$\lambda_{i-\text{rk}(R_n)}(A_n) \le \lambda_i(A_n + R_n) \le \lambda_{i+\text{rk}(R_n)}(A_n)$$

- Weyl Theorem

$$|\lambda_i(A_n) - \lambda_i(A_n + N_n)| \le \|N_n\|$$

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Counterexample

$$A_n = \frac{1}{n}\begin{pmatrix} & 1 & & & \\ 1 & & 1 & & \\ & 1 & & \ddots & \\ & & \ddots & & 1 \\ & & & 1 & \end{pmatrix} \sim_\lambda 0 \qquad R_n = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ n^{n-1} & & & & \end{pmatrix}$$

$$A_n + R_n \sim \begin{pmatrix} & 1 & & & \\ n^{-2} & & 1 & & \\ & n^{-2} & & \ddots & \\ & & \ddots & & 1 \\ 1 & & & n^{-2} & \end{pmatrix} \sim_\lambda e^{2\pi i x}$$

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Non-Hermitian Perturbation

Theorem [Golinskii-Serra '07]

Let X_n Hermitian matrix of size n, with $\{X_n\}_n \sim_\lambda k$. If

- $\|Y_n\|_1 = o(n)$
- $\|Y_n\|, \|X_n\| = O(1)$

then

$$\{X_n + Y_n\}_n \sim_\lambda k$$

CDR Equations

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), & x \in (0,1), \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases}$$

$a(x), b(x), c(x)$ bounded a.e. continuous functions

$$\xrightarrow{FD} \{A_n\}_n + \{B_n\}_n + \{C_n\}_n \sim_\lambda a(x)(2 - 2\cos(\theta)).$$

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Conjecture

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Theorem [Barbarino-Serra ’17]

Let X_n Hermitian matrix of size n, with $\{X_n\}_n \sim_\lambda k$. If

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Main Result

Theorem [Barbarino-Serra '17]

Let $\{X_n\}_n$ be an Hermitian sequence with spectral symbol $\{X_n\}_n \sim_\lambda k$. If

$$\|Y_n\|_2 = o(\sqrt{n})$$

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$$\{X_n\}_n + \{Y_n\}_n \sim_\lambda k$$

Corollary

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Preliminary Results

Theorem

If A, B are Hermitian matrices with eigenvalues $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, then

$$\sum_{i=1}^{n} |\alpha_i - \beta_i|^2 \leq \|A - B\|_2^2$$

$\downarrow$ Real/Imaginary Part
Schur Decomposition

Lemma

If A is an Hermitian matrices with eigenvalues $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$, B is any matrix with eigenvalues $\Re(\beta_1) \geq \Re(\beta_2) \geq \cdots \geq \Re(\beta_n)$, then

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Proof of Theorem

Let $\{X_n\}_n \sim_\lambda k$ be an Hermitian sequence and $\|Y_n\|_2 = o(\sqrt{n})$

- Let $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$ be the eigenvalues of X_n and $\Re(\beta_1) \geq \Re(\beta_2) \geq \dots \geq \Re(\beta_n)$ be the eigenvalues of $X_n + Y_n$.
- Let $k_{n,\varepsilon} := \#\{i : |\alpha_i - \beta_i| > \varepsilon\}$
- $$\frac{k_{n,\varepsilon}}{n}\varepsilon^2 \leq \frac{\sum_{i=1}^n |\alpha_i - \beta_i|^2}{n} \leq \frac{\|Y_n\|_2^2}{n} \to 0$$

$$f(X_n + Y_n) - f(X_n) \xrightarrow{\mu} 0, \quad f(X_n) \xrightarrow{\mu} k \implies f(X_n + Y_n) \xrightarrow{\mu} k$$

$$\{X_n + Y_n\}_n \sim_\lambda k$$

Proof of Theorem

Let $\{X_n\}_n \sim_\lambda k$ be an Hermitian sequence and $\|Y_n\|_2 = o(\sqrt{n})$

- Let $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n$ be the eigenvalues of X_n and $\Re(\beta_1) \geq \Re(\beta_2) \geq \dots \geq \Re(\beta_n)$ be the eigenvalues of $X_n + Y_n$.
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Applications

CDR Equations

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in (0,1), \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases}$$

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FD

- $a(x), c(x)$ bounded a.e. continuous functions
- $b(x)$ bounded a.e. continuous function

$$\xrightarrow{FD} \{A_n\}_n + \{B_n\}_n + \{C_n\}_n \sim_\lambda a(x)(2 - 2\cos(\theta)),$$

Minimal Hypothesis

$$b(x) \sim x^\alpha \implies \|B_n\|_2 = O(n^{-1-\alpha})$$

$$\alpha > -\frac{3}{2} \implies \|B_n\|_2 = o(\sqrt{n}) \qquad \alpha > -2 \implies \|B_n\|_2 = o(n)$$

Applications

CDR Equations

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), & x \in (0,1), \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases}$$

FD

- $a(x), c(x)$ ~~bounded~~ a.e. continuous functions
- $b(x)$ integrable continuous function

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FE

- $a(x), b(x), c(x)$ bounded measurable functions

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FE

- $a(x), b(x), c(x)$ integrable functions

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Preconditioned

- $a(x), c(x)$ bounded a.e. continuous functions
- $b(x)$ bounded a.e. continuous function
- $\{K_n\}_n \sim_\lambda \phi$ sequence of definite positive Hermitian matrices

$$\xrightarrow{PrecFD} \{K_n^{-1}(A_n + B_n + C_n)\}_n \sim_\lambda \frac{a(x)(2 - 2\cos(\theta))}{\phi(x)},$$

$$K_n = T_n(2 - 2cos(\theta)) \sim_\lambda 2 - 2cos(\theta)$$

$$K_n = D_n(a^{1/2})T_n(2 - 2cos(\theta))D_n(a^{1/2}) \sim_\lambda a(x)(2 - 2cos(\theta))$$

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Applications

CDR Equations

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\begin{cases}
-\nabla \cdot \mathrm{A}\nabla u + \mathbf{b} \cdot \nabla u + cu = f, & \text{in } (0,1)^d, \\
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\end{cases}
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Applications

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d-dimensional FD

- $A(\mathbf{x})$ symmetric matrix of $C^1(0,1)^d$ bounded functions with bounded partial derivatives
- $\mathbf{b}(\mathbf{x})$ bounded continuous function
- $c(\mathbf{x})$ bounded continuous function

$$\xrightarrow{FD} \{A_{\boldsymbol{n}} + B_{\boldsymbol{n}} + C_{\boldsymbol{n}}\}_n \sim_\lambda \mathbf{1}(\mathrm{A}(\mathbf{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T.$$

Minimal Hypothesis

$$\mathbf{b}(\mathbf{x}) \sim \mathbf{x}^\alpha \implies \|B_{\boldsymbol{n}}\|_2 = O(n^{-1-d\alpha})$$

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Numerical Examples

Coefficients with $o(\sqrt{n})$ Perturbation

We consider the CDR equations with...

- $a(x) = c(x) = -\log(1-x)$, $b(x) = 1/\sqrt[4]{x^5}$ for FD and FE discretizations
- $a(x) = c(x) = -\log(1-x)$, $b(x) = 1/\sqrt[4]{x^3}$ and $K_n = T_n(2 - 2cos(\theta))$ in the preconditioned discretization
- $a_{1,1}(x,y) = c(x,y) = 1/xy$, $a_{2,2}(x,y) = -xy$, $a_{1,2}(x,y) = x + y$, $b_1(x,y) = b_2(x,y) = 1/\sqrt[4]{(xy)^3}$ for a bidimensional FD discretization

And we analyse...

- Number of eigenvalues of $A_n + B_n + C_n$ with imaginary part greater than ε
- The graph of the real part of eigenvalues of $A_n + B_n + C_n$ against the symbol

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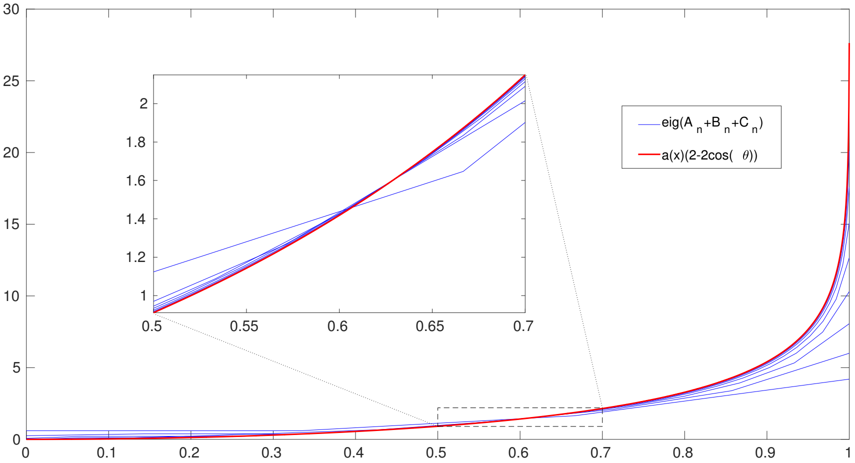
Offliers

N		50	100	200	400	800
FD-1-dim	$\varepsilon = 10^{-1}$	4/8%	6/6%	10/5%	12/3%	14/1.75%
	$\varepsilon = 10^{-2}$	6/12%	8/8%	14/7%	20/5%	28/3.5%
FE	$\varepsilon = 10^{-1}$	4/8%	6/6%	8/4%	12/3%	14/1.75%
	$\varepsilon = 10^{-2}$	4/8%	8/8%	12/6%	14/4.5%	26/3.25%
FD-2-dim	$\varepsilon = 10^{-1}$	4/8.16%	2/2%	2/1.02%	2/0.5%	2/0.26%
	$\varepsilon = 10^{-2}$	8/16.33%	12/12%	30/15.31%	52/13%	92/11.74%
Prec	$\varepsilon = 10^{-1}$	2/4%	2/2%	2/1%	2/0.5%	4/0.5%
	$\varepsilon = 10^{-2}$	6/12%	8/8%	10/5%	12/3%	16/2%

Table 1: *Number and percentage of eigenvalues with imaginary part greater than* ε.

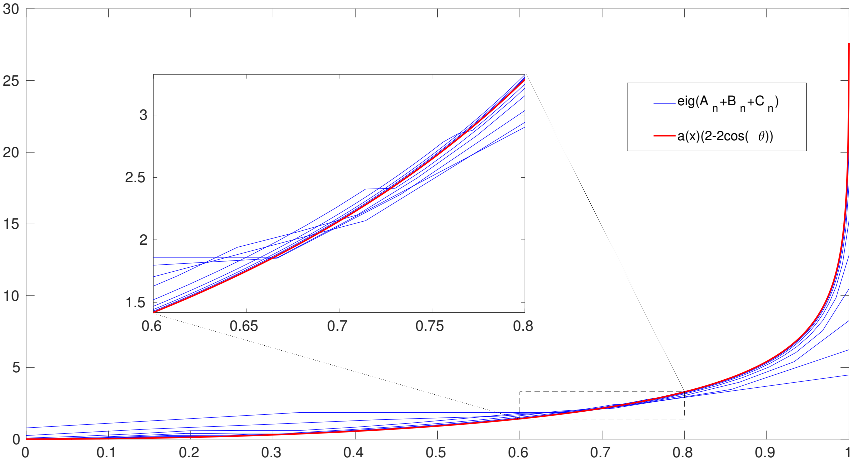
Eigenvalues Graph

FD



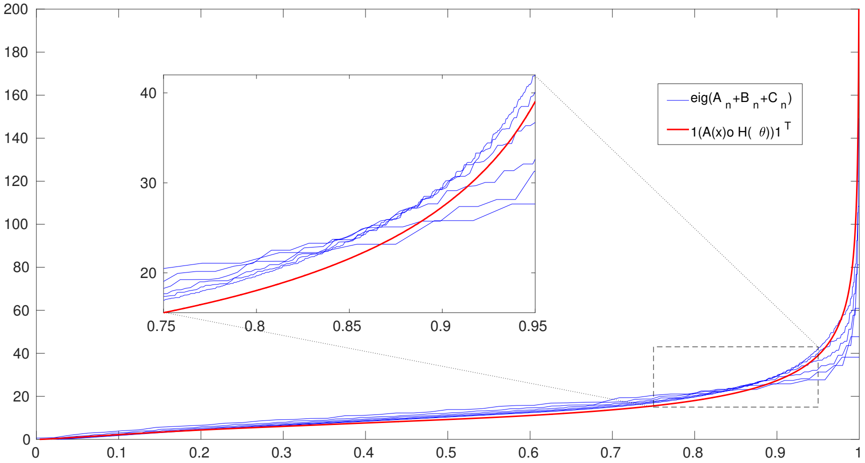
Eigenvalues Graph

FE



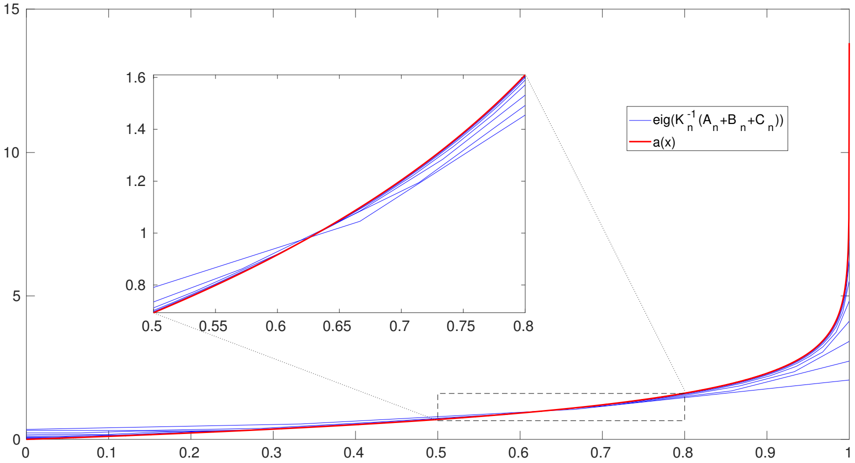
Eigenvalues Graph

2-dimension FD



Eigenvalues Graph

Preconditioned



Coefficients with $o(n)$ Perturbation

We consider the CDR equations with...

- $a(x) = c(x) = -\log(1-x)$, $b(x) = 1/\sqrt[4]{x^7}$ for FD and FE discretizations
- $a_{1,1}(x,y) = a_{2,2}(x,y) = 1/\sqrt{xy}$, $c(x,y) = 1/xy$, $a_{1,2}(x,y) = x + y$, $b_1(x,y) = b_2(x,y) = 1/\sqrt[4]{(xy)^5}$ for a bidimensional FD discretization

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Offliers

N		50	100	200	400	800
FD-1-dim	$\varepsilon = 10^{-1}$	8/16%	12/12%	18/9%	28/7%	40/5%
	$\varepsilon = 10^{-2}$	8/16%	14/14%	22/11%	34/8.5%	74/9.25%
FE	$\varepsilon = 10^{-1}$	6/12%	12/12%	18/9%	26/6.5%	40/5%
	$\varepsilon = 10^{-2}$	4/8%	8/8%	12/6%	14/4.5%	26/3.25%
FD-2-dim	$\varepsilon = 10^{-1}$	12/24.29%	20/20%	30/15.31%	44/11%	58/7.4%
	$\varepsilon = 10^{-2}$	16/32.653%	24/24%	42/21.43%	64/16%	114/14.54%

Table 2: *Number and percentage of eigenvalues with imaginary part greater than ε.*

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