Symbols for matrix-sequences: Application-Driven Structure

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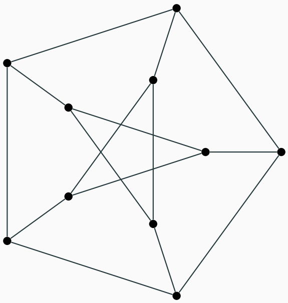
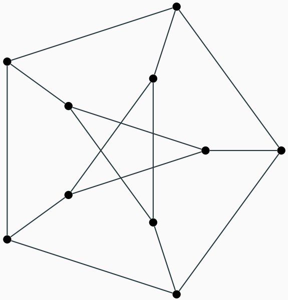
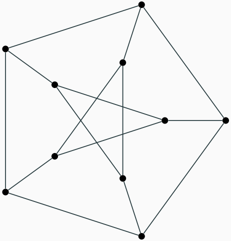


Petersen Graphs



$GPG(5, 2)$

Petersen Graphs



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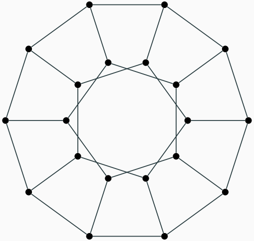
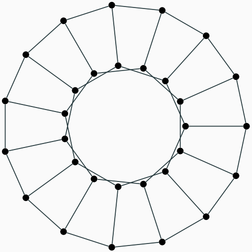


GPG(10,2)



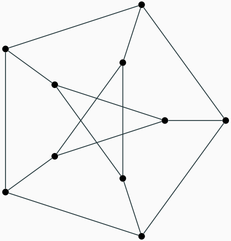
GPG(15,2)

$\downarrow$ $\downarrow$ $\downarrow$

A_5 A_{10} A_{15}

$$\{GPG(n,2)\}_n \longrightarrow \{A_n\}_n$$

Petersen Graphs



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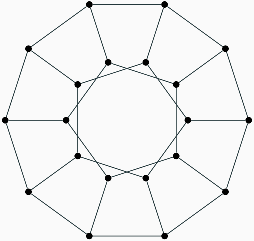

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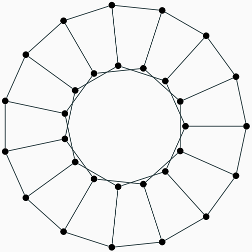
A_{10}



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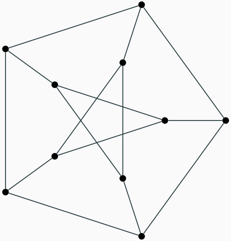

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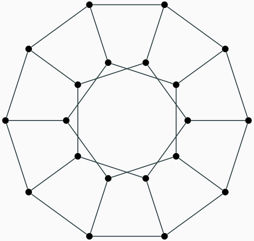

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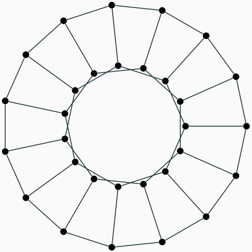

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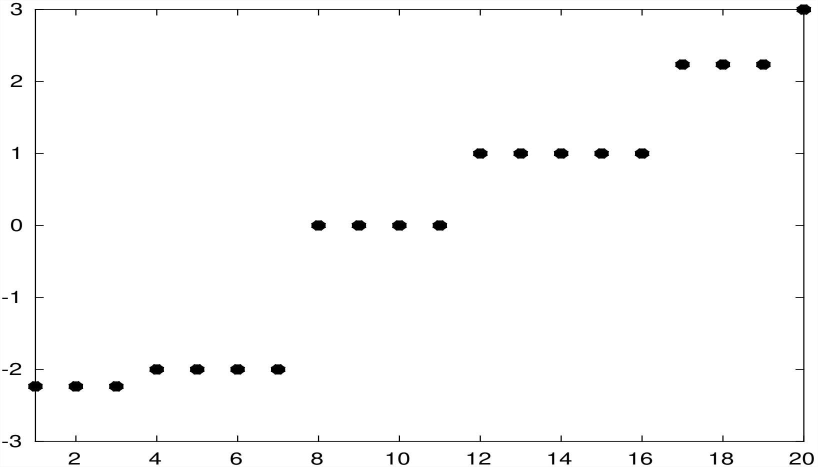

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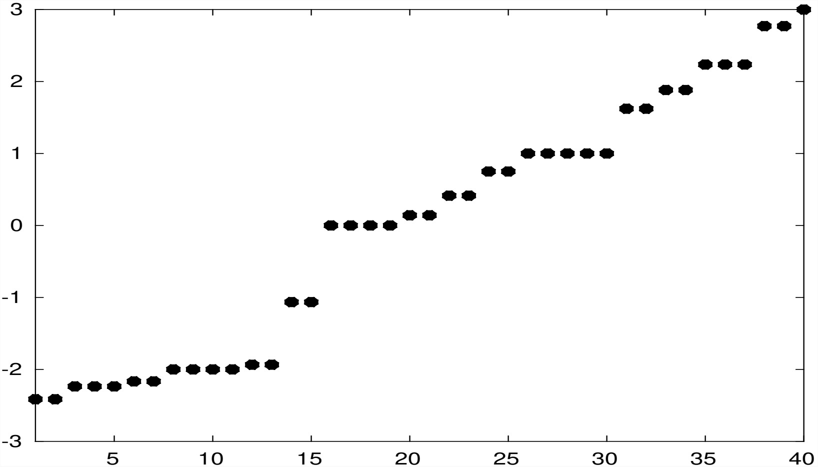
$$n = 20$$



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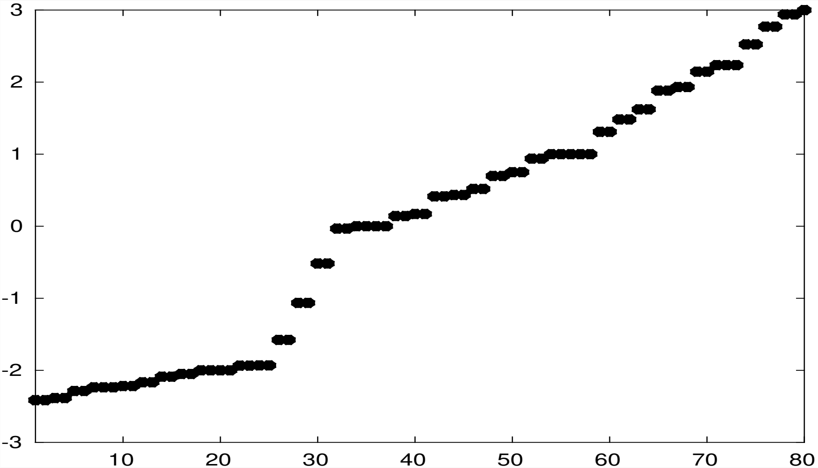
$n = 40$



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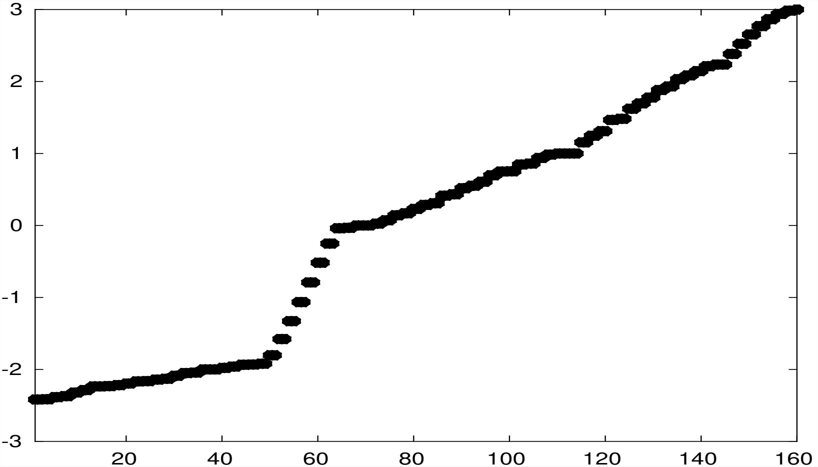
$n = 80$



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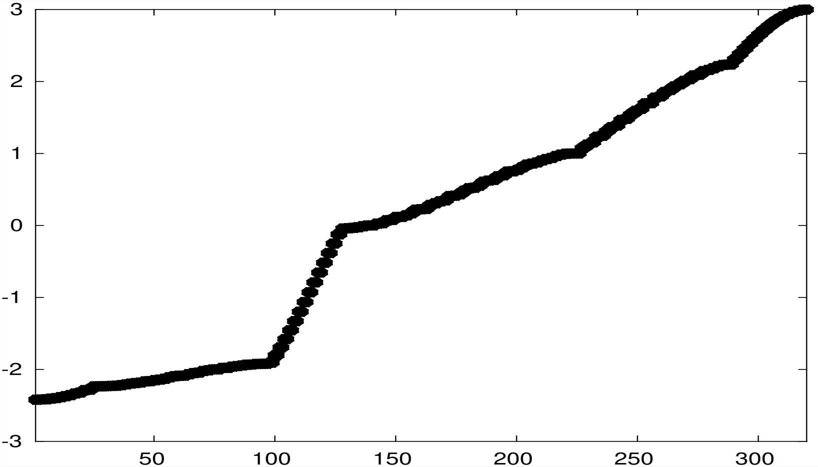
$n = 160$



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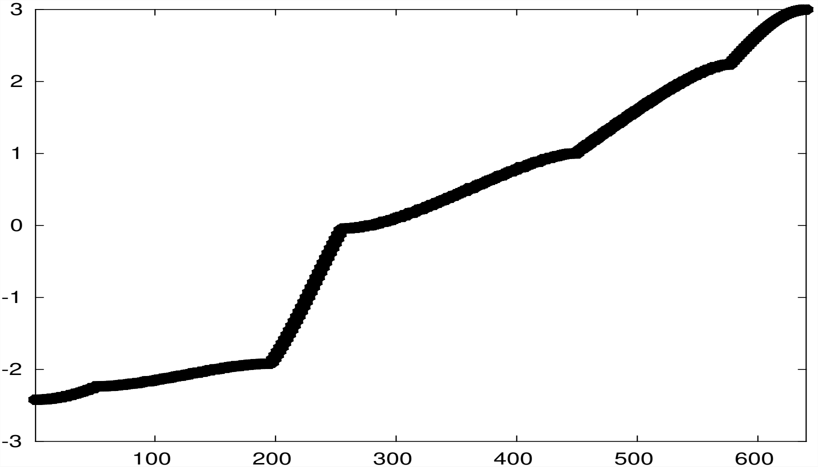

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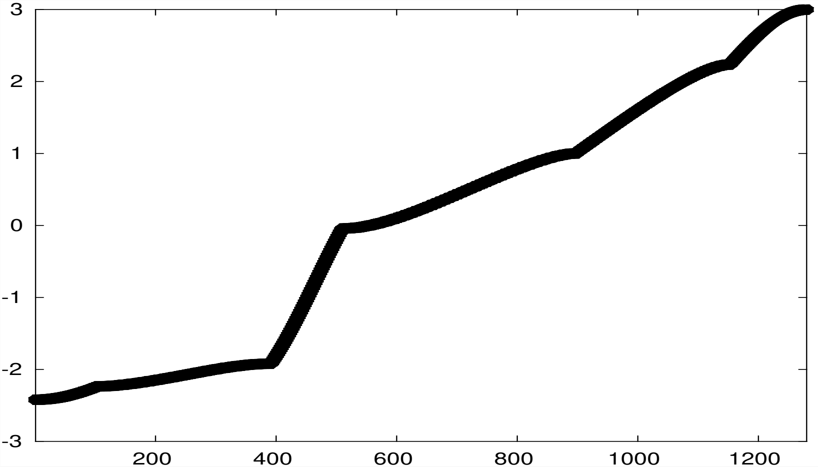
$n = 640$



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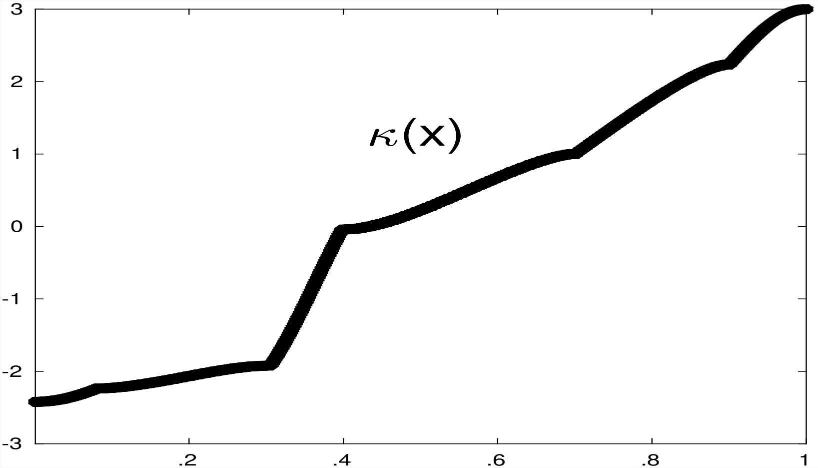
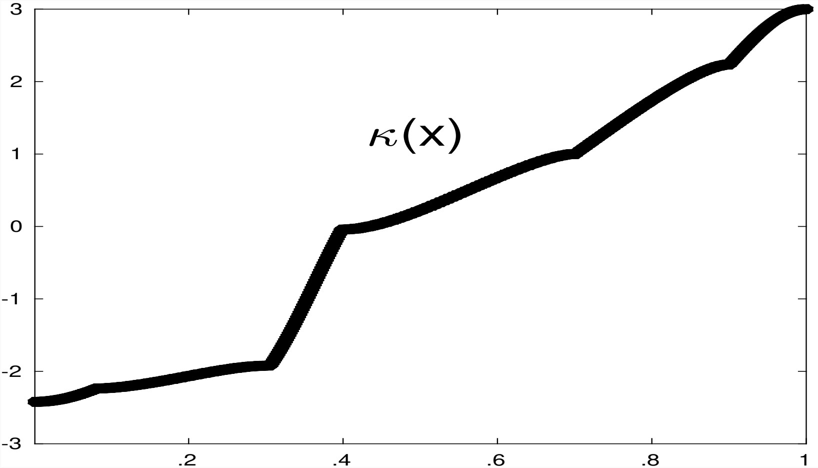


Petersen Graphs

$$\{A_n\}_n \sim \kappa(x)$$



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$$A_n u_n = f_n \xrightarrow[\text{Quasi-Newton, CG}]{\text{Preconditioned Krylov}} u_n$$

$$\uparrow$$

$$\Lambda(A_n)$$

Prior informations on the eigenvalues let us choose the best couple of discretization/solver for the PDE

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Simple Example

$$\begin{cases} -u''(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \quad \xrightarrow{FD} \quad A_n u_n = f_n$$

$$A_n = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\lambda_j(A_n) = 2 - 2\cos\left(\frac{j\pi}{n+1}\right) \qquad \kappa(t) = 2 - 2\cos(t)$$

$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n \\ \lambda_{2(n+1)-2j}(A_n), & 2j > n \end{cases} \qquad \widetilde{\kappa}(t) = 2 - 2\cos(2t)$$

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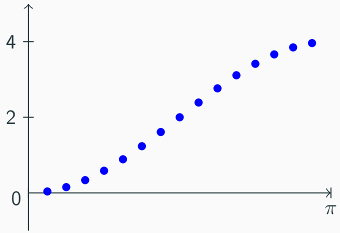

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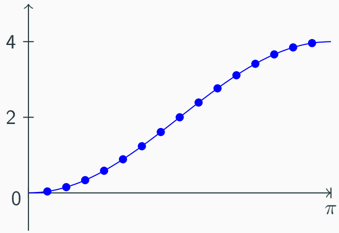

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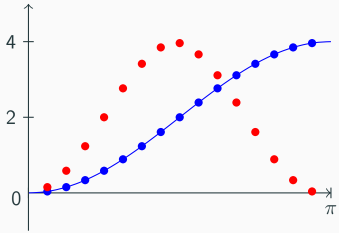

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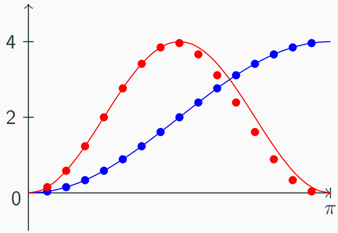

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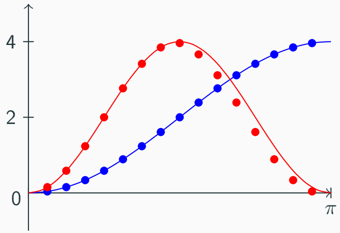

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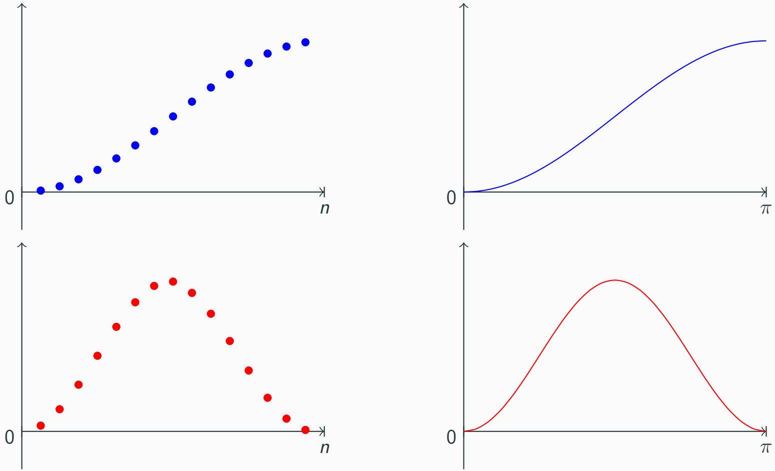

$$\frac{\#\{i : a < \lambda_i(A_n) < b\}}{n} \xrightarrow{n\to\infty} \frac{\mu\{t : a < \kappa(t) < b\}}{\pi}$$

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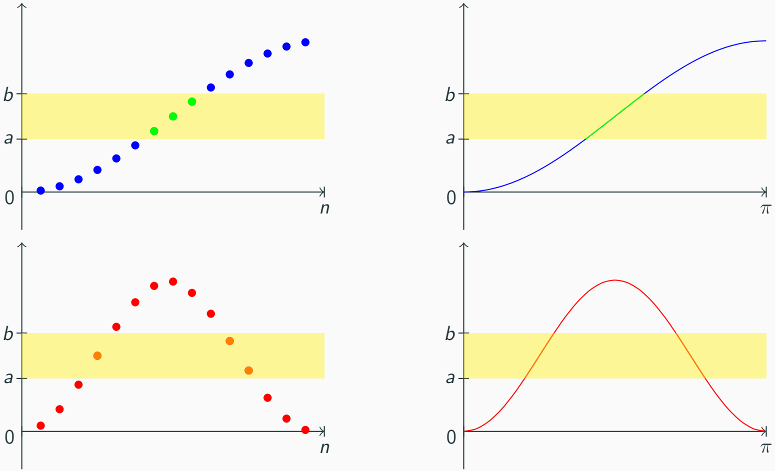

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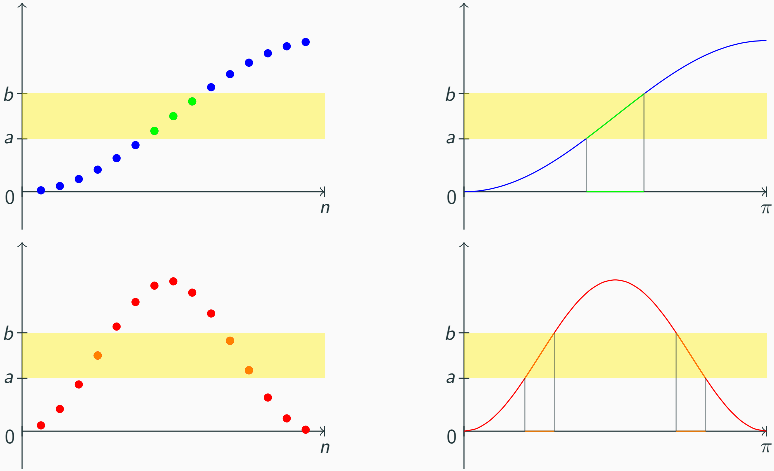

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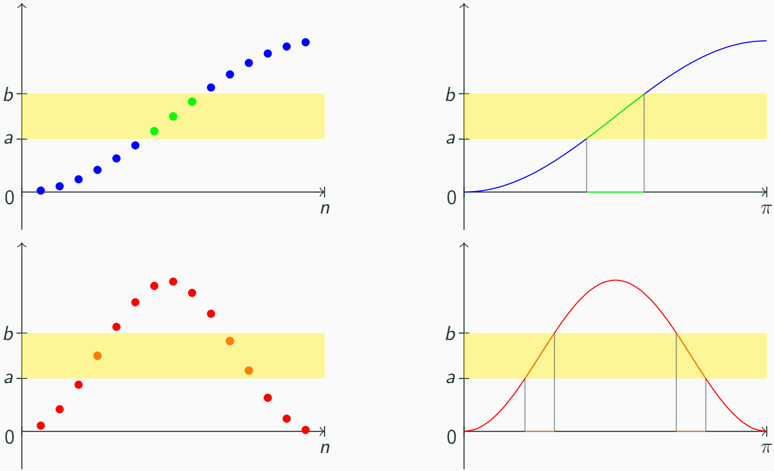

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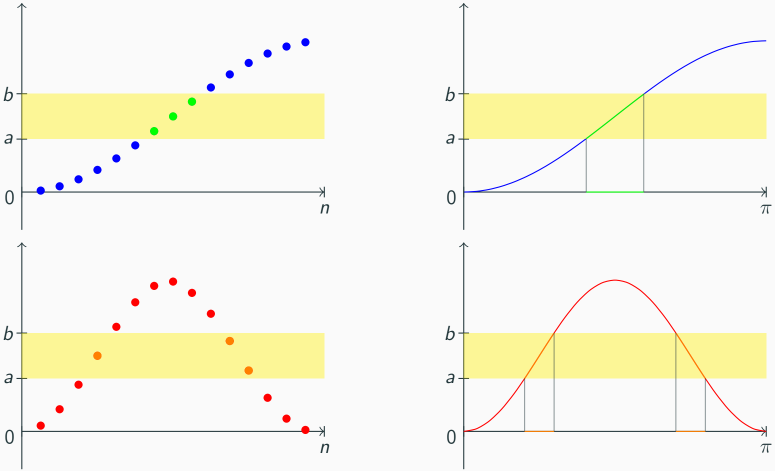

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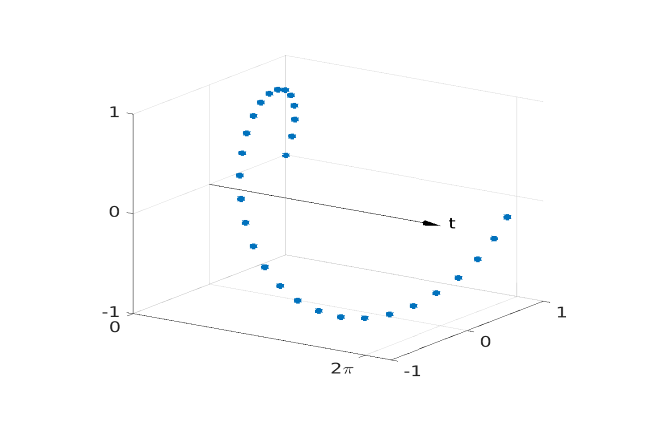

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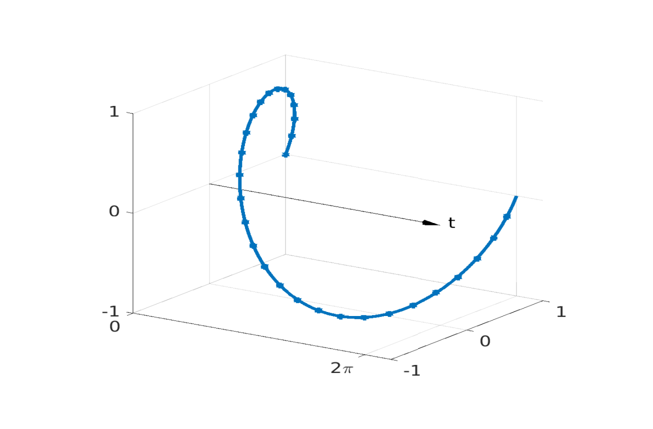
$$C_n = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \quad \longrightarrow \quad \{C_n\}_n \sim e^{t\mathrm{i}}, \quad t \in [0, 2\pi]$$



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Sturm-Liouville

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$$\{A_n\}_n \sim ?$$

Sturm-Liouville

$$\begin{cases} -(a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

$$a_k := a\left(\frac{k}{2(n+1)}\right)$$

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- $D_n = \mathrm{diag}(a(i/n))_{i=1,\ldots,n}$
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- $S_{n^2} = D_n \otimes T_n$

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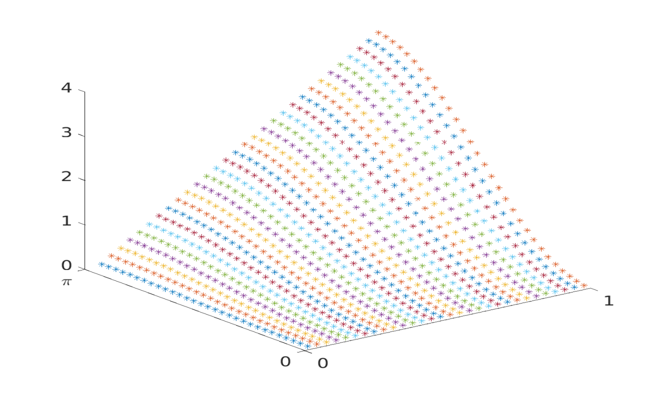

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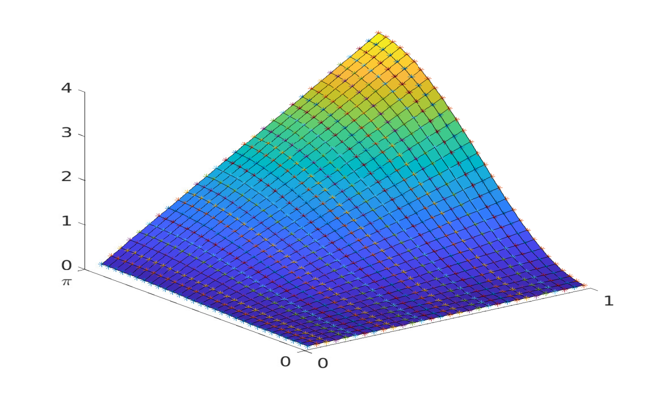

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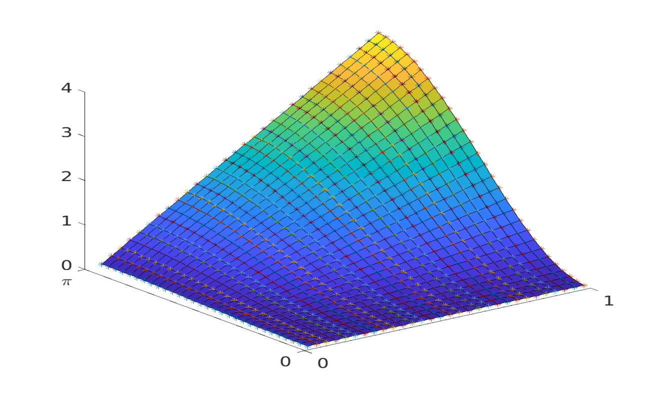

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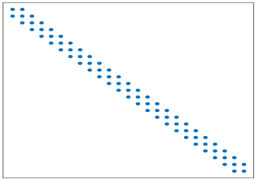
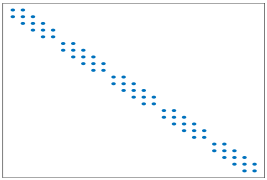


S_n



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$$A_n \qquad\qquad S_n$$

$$\begin{pmatrix} \ddots & \ddots & \\ a\left(\frac{2k+1}{2(n+1)}\right) & a\left(\frac{2k+1}{2(n+1)}\right) + a\left(\frac{2k+3}{2(n+1)}\right) & a\left(\frac{2k+3}{2(n+1)}\right) \\ & \ddots & \ddots \end{pmatrix} \qquad \begin{pmatrix} \ddots & \ddots & \\ a\left(\frac{\lfloor k/\sqrt{n}\rfloor}{\sqrt{n}}\right) & 2a\left(\frac{\lfloor k/\sqrt{n}\rfloor}{\sqrt{n}}\right) & a\left(\frac{\lfloor k/\sqrt{n}\rfloor}{\sqrt{n}}\right) \\ & \ddots & \ddots \end{pmatrix}$$

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$$A_n - S_n = R_n + N_n \qquad \frac{\mathrm{rk}(R_n)}{n} \to 0 \quad \|N_n\| \to 0$$

$$\{S_n\}_n \sim a(x)(2 - 2\cos(\theta)) \iff \{A_n\}_n \sim a(x)(2 - 2\cos(\theta))$$

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GLT Sequences

Special Sequences

- $\{Z_n\}_n \sim 0 \to \mathcal{Z} = \{(\{Z_n\}_n, 0)\}$
- $\{D_n(a)\}_n \sim a(x) \to \mathcal{D} = \{(\{D_n(a)\}_n, a(x))\}$
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$$a \in C[0,1]$$

$$D_n(a) := \begin{pmatrix} a(1/n) & & & & \\ & a(2/n) & & & \\ & & a(3/n) & & \\ & & & \ddots & \\ & & & & a(1) \end{pmatrix}$$

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$$f \in L^1[-\pi, \pi] \to \widehat{f_n} = \int_{-\pi}^{\pi} f(\theta) e^{-\mathrm{i}n\theta} d\theta$$

$$T_n(f) := \begin{pmatrix} \widehat{f_0} & \widehat{f_1} & \widehat{f_2} & \dots & \widehat{f_{n-1}} \\ \widehat{f_{-1}} & \widehat{f_0} & \ddots & \ddots & \vdots \\ \widehat{f_{-2}} & \ddots & \ddots & \ddots & \widehat{f_2} \\ \vdots & \ddots & \ddots & \widehat{f_0} & \widehat{f_1} \\ \widehat{f_{-n+1}} & \dots & \widehat{f_{-2}} & \widehat{f_{-1}} & \widehat{f_0} \end{pmatrix}$$

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$$\mathcal{G} := \mathbb{C}[\mathcal{Z}, \mathcal{D}, \mathcal{T}] \qquad \text{(GLT space)}$$

Theorem

$$(\{A_n\}_n, \kappa(x,\theta)) \in \mathcal{G} \implies \{A_n\}_n \sim \kappa(x,\theta)$$

$$\begin{cases} (a(x)u'(x))' = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \xrightarrow{FD} A_n u_n = f_n$$

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Applications

- $(a(x)u'(x))' = f(x) \qquad x \in [0,1]$

  $\xrightarrow{FD} a(x)(2 - 2\cos(\theta)) \qquad (x,\theta) \in [0,1] \times [-\pi,\pi]$

- $(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \qquad x \in [0,1]$

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- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{IGA} \mathbf{1}(A_G(\hat{x}) \circ H_p(\theta))\mathbf{1}^T \qquad (\hat{x}, \theta) \in \hat{\Omega} \times [-\pi, \pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(\hat{x}))}{G'(\hat{x})^2} h_p(\theta) \right)$$

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$$\xrightarrow{IgA\ Gal., Coll.(p)} \mathbf{1}(A(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in \Omega$, irregular grid

$$\xrightarrow{...(G)} \mathbf{1}(A_G(\boldsymbol{x}) \circ H_p(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in \Omega' \times [-\pi, \pi]^d$$

$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} h_p(\theta) \right)$$

Applications

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

$$\xrightarrow{FD, P1-FE} \mathbf{1}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\mathbf{1}^T \qquad (\boldsymbol{x}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$$

- $-\nabla \cdot \mathrm{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \boldsymbol{x} \in [0,1]^d$

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Applications

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$$\xrightarrow{d=1} \left(\frac{a(G(x))}{G'(x)^2} f_p(\theta) \right)$$

If there's time..

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$$
\begin{cases}
-a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x) & x \in [0,1] \\
\quad a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x) & x \in [0,1]
\end{cases}
$$

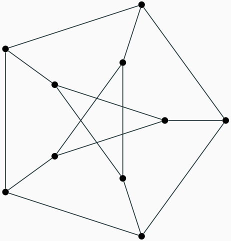
If there's time..

$$\begin{cases} -a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x) & x \in [0,1] \\ a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x) & x \in [0,1] \end{cases}$$

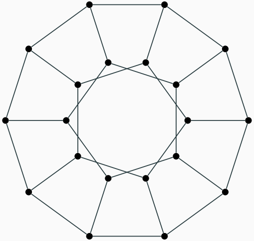
GPG(5,2) GPG(10,2) GPG(15,2)

If there's time..

$$\begin{cases} -a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x) & x \in [0,1] \\ a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x) & x \in [0,1] \end{cases}$$



GPG(5,2)



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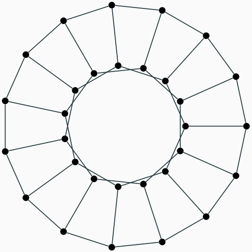

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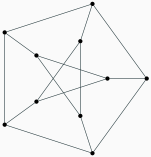

$$\rightsquigarrow \begin{pmatrix} \mathrm{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \mathrm{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \mathrm{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi(k-1)}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi(k-1)}{n}\right) \end{pmatrix}\right)$$

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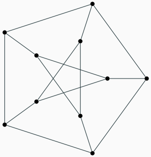
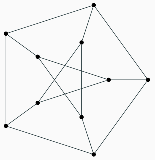


$$A_5 = \begin{pmatrix} C_5 + C_5^T & I_5 \\ I_5 & C_5^2 + (C_5^2)^T \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi(k-1)}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi(k-1)}{n}\right) \end{pmatrix}\right)$$

If there's time..



$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

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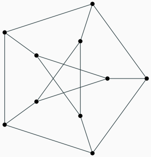
$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$



$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

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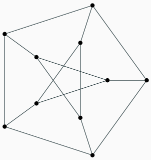
$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$



$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi(k-1)}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi(k-1)}{n}\right) \end{pmatrix}\right)$$

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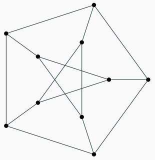
$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$



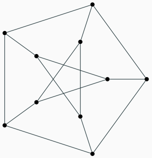
$$\rightsquigarrow \begin{pmatrix} \operatorname{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \operatorname{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

$$\rightsquigarrow \operatorname{diag}\left(\begin{pmatrix} 2\cos\left(\frac{2\pi(k-1)}{n}\right) & 1 \\ 1 & 2\cos\left(\frac{4\pi(k-1)}{n}\right) \end{pmatrix}\right)$$

$$\begin{aligned} \lambda_{k,1}(A_n) &= \cos((k-1)2\pi/n) + \cos((k-1)4\pi/n) \\ &\quad + \sqrt{[\cos((k-1)2\pi/n) - \cos((k-1)4\pi/n)]^2 + 1} \\ \lambda_{k,2}(A_n) &= \cos((k-1)2\pi/n) + \cos((k-1)4\pi/n) \\ &\quad - \sqrt{[\cos((k-1)2\pi/n) - \cos((k-1)4\pi/n)]^2 + 1} \end{aligned}$$

If there's time..

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$



$$\rightsquigarrow \begin{pmatrix} \mathrm{diag}\left(2\cos\left(\frac{2\pi(k-1)}{n}\right)\right) & I_n \\ I_n & \mathrm{diag}\left(2\cos\left(\frac{4\pi(k-1)}{n}\right)\right) \end{pmatrix}$$

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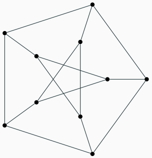
$$\lambda_1(x) = \cos(2\pi x) + \cos(4\pi x) + \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

$$\lambda_2(x) = \cos(2\pi x) + \cos(4\pi x) - \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

If there's time..

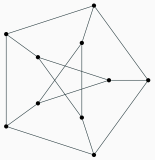


$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

$$\begin{aligned} \lambda_1(x) &= \cos(2\pi x) + \cos(4\pi x) \\ &\quad + \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1} \\ \lambda_2(x) &= \cos(2\pi x) + \cos(4\pi x) \\ &\quad - \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1} \end{aligned}$$

If there's time..

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$



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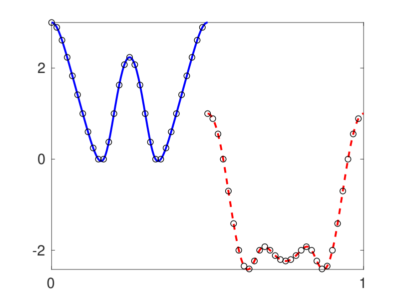
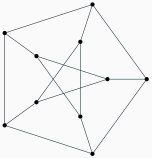
$$\lambda_2(x) = \cos(2\pi x) + \cos(4\pi x) - \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

$$\widetilde{\kappa}(x) = \begin{cases} \lambda_1(2x) & x \in [0, 1/2) \\ \lambda_2(2x-1) & x \in [1/2, 1] \end{cases}$$



If there's time..



$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} T_n(2\cos(\theta)) & I_n \\ I_n & T_n(2\cos(2\theta)) \end{pmatrix}$$

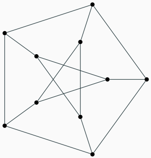
$$\sim \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(2\theta) \end{pmatrix}$$

$$\lambda_1(x) = \cos(2\pi x) + \cos(4\pi x) + \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

$$\lambda_2(x) = \cos(2\pi x) + \cos(4\pi x) - \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

$$(2\pi x \to \theta)$$

If there's time..



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$$(2\pi x \to \theta)$$

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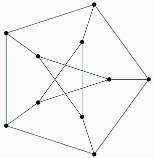

$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

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$$\lambda_1(x) = \cos(2\pi x) + \cos(4\pi x) + \sqrt{[\cos(2\pi x) - \cos(4\pi x)]^2 + 1}$$

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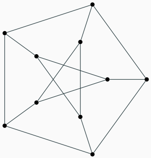
$$(2\pi x \to \theta)$$

If there's time..



$$A_n = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^2 + (C_n^2)^T \end{pmatrix}$$

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$$(2\pi x \to \theta)$$

If there's time..

$$\begin{cases} -a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x) & x \in [0,1] \\ a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x) & x \in [0,1] \end{cases}$$

$$A_n = \begin{pmatrix} M_n & N_n \\ P_n & Q_n \end{pmatrix} \sim \begin{pmatrix} \kappa_{11}(x,\theta) & \kappa_{12}(x,\theta) \\ \kappa_{21}(x,\theta) & \kappa_{22}(x,\theta) \end{pmatrix}$$

$$\Lambda(A_n) \sim \Lambda \begin{pmatrix} \kappa_{11}(x,\theta) & \kappa_{12}(x,\theta) \\ \kappa_{21}(x,\theta) & \kappa_{22}(x,\theta) \end{pmatrix}$$

If there's time..

$$\begin{cases} -a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x) & x \in [0,1] \\ a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x) & x \in [0,1] \end{cases}$$

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That's All, Folks!