The Fast Resampled Iterative Filtering Method

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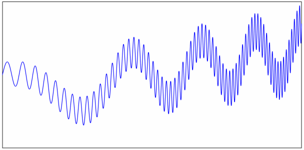
Iterative Filtering

Empirical Method Decomposition (EMD)

Signal $s(x)$



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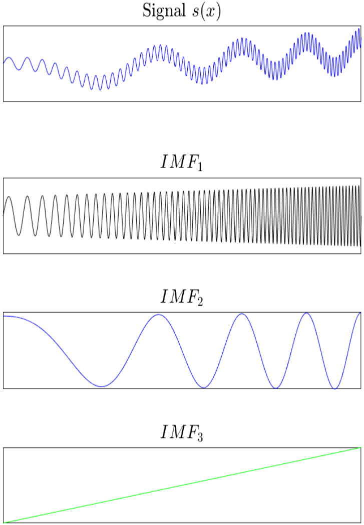

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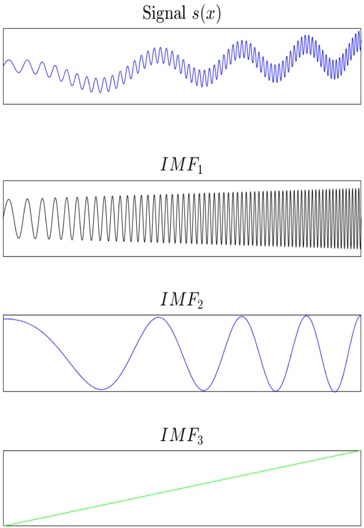
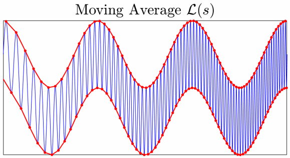
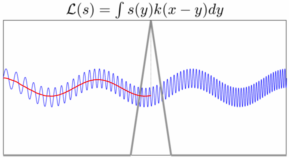


The effect of the moving average is to flatten the highest frequency component



A way to emulate the effect is to use a filter on the signal



Iterative Filtering

Choose the filter k:

- Unit-norm, even, nonnegative and compact supported
- $k = \omega \star \omega$

$$\implies 0 \leq \hat{k}(\xi) \leq 1$$

The IF method iteratively apply the filter through convolution

$$\mathcal{S}(f) := f(x) - \int f(y)k(x-y)dy$$
$$IMF = IMF \cup \{\mathcal{S}^{\infty}(s)\}$$
$$s = s - \mathcal{S}^{\infty}(s)$$

On the Time Dimension the Sifting Operator is the difference between the signal and the Moving Average and it extracts the higher frequencies.

This and the convergence of $\mathcal{S}^{\infty}(s)$ can be studied on the frequencies space

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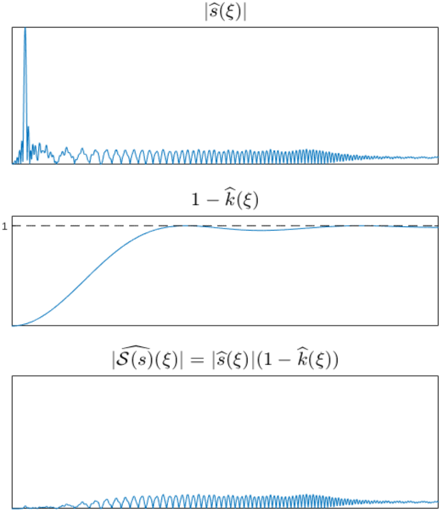
On the Frequency Domain

$$\widehat{\mathcal{S}(s)}(\xi) = \widehat{s}(\xi)(1 - \widehat{k}(\xi))$$
$$\widehat{\mathcal{S}^m(s)}(\xi) = \widehat{s}(\xi)(1 - \widehat{k}(\xi))^m$$



The Fundamental Zero and the Stopping Criterion

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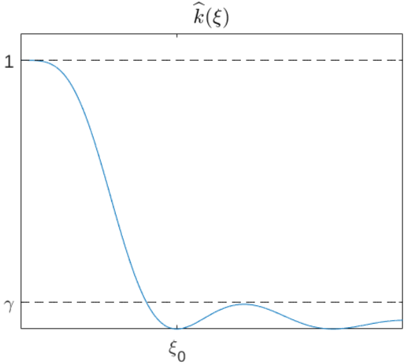

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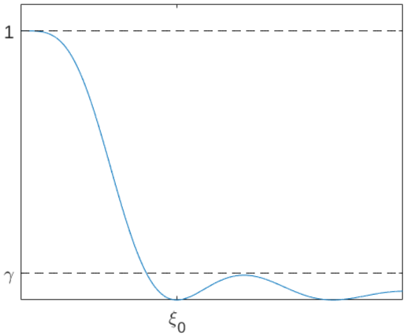

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Theorem (B. 2023)

If we choose ξ_0 depending on the biggest frequency in $\widehat{s}$ whose intensity is at least η, then

$$B(\xi_0, C \sqrt[2p]{\eta\delta}) \subseteq J_0$$

where $2p$ is the order of ξ_0 the first zero in $\widehat{k}$, and δ depends on the stopping criterion

Bigger J_0 achieves better decomposition, especially for amplitude-modulated signals

$$s(x) = a(x)g(x) \implies \widehat{s}(\xi) = (\widehat{a} \star \widehat{g})(\xi)$$

where $a(x)$ has low instant frequency

Discrete Setting

The signal $s(x)$ is studied on $[0,1]$ and it is supposed to be periodic at the boundaries [**Stallone, Cicone, Materassi 2020**] so that the discretization results in a circulant matrix

$$\boldsymbol{s} = [s(h)\ s(2h)\ \dots\ s(1-h)\ s(1)] \qquad h = 1/N$$

$$\mathcal{S}(s)(x) = s(x) - \int_0^1 s(x-y)k(y)dy|_{x=ah} \qquad \mathcal{S}(s)(ah) \sim \boldsymbol{s}_a - \frac{1}{N}\sum_{b=1}^{N} k\,(bh)\,\boldsymbol{s}_{a-b}$$

$$\mathcal{S}(\boldsymbol{s}) := \boldsymbol{s} - K\boldsymbol{s} = (I-K)\boldsymbol{s}$$

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One can thus write the main loop of the discrete IF Algorithm as

$$\mathcal{S}(\boldsymbol{f}) := (I-K)\boldsymbol{f}$$
$$\boldsymbol{IMF} = \boldsymbol{IMF} \cup \{\mathcal{S}^m(\boldsymbol{s})\}$$
$$\boldsymbol{s} = \boldsymbol{s} - \mathcal{S}^m(\boldsymbol{s})$$

where the stopping condition is $\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta$

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Fast IF

$$\mathcal{S}^m(\boldsymbol{s}) = (I-K)^m\boldsymbol{s} \implies \widehat{\mathcal{S}^m(\boldsymbol{s})} = \boldsymbol{k}^{\circ m} \circ \widehat{\boldsymbol{s}}$$

where $\boldsymbol{k}$ is the first row of $I-K$, $\circ$ is the elementwise product and $\widehat{\boldsymbol{s}}$ is the DFT of $\boldsymbol{s}$

$$\|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta \iff \|\boldsymbol{k}^{\circ m} \circ (\boldsymbol{k} - \boldsymbol{e}) \circ \widehat{\boldsymbol{s}}\| < \delta$$

The stopping condition can be checked on $\boldsymbol{k}$ and $\widehat{\boldsymbol{s}}$ with linear cost + 2 DFT per IMF

Theorems in the Discrete Settings

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Theorem (Cicone, Zhou, 2021, B. 2023)

Given $\delta > 0$, $\boldsymbol{s}$, *then*

$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|\boldsymbol{s}\|}$$

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For any vectors $\boldsymbol{h}, \boldsymbol{s}$ let K be any $n \times n$ Hermitian matrix with spectrum in $[0, 1]$. Then

$$\|\mathcal{S}^m(\boldsymbol{s} + \boldsymbol{h}) - \mathcal{S}^m(\boldsymbol{s})\| \leq \|\boldsymbol{h}\|.$$

If now the filters and m_j are fixed, for $\boldsymbol{IMF}_j$ the modes generated by $\boldsymbol{s}$ and for $\boldsymbol{IMF}_j^*$ generated by $\boldsymbol{s} + \boldsymbol{h}$, we have

$$\sum_j \|\boldsymbol{IMF}_j^* - \boldsymbol{IMF}_j\|^2 \leq \|\boldsymbol{h}\|^2.$$

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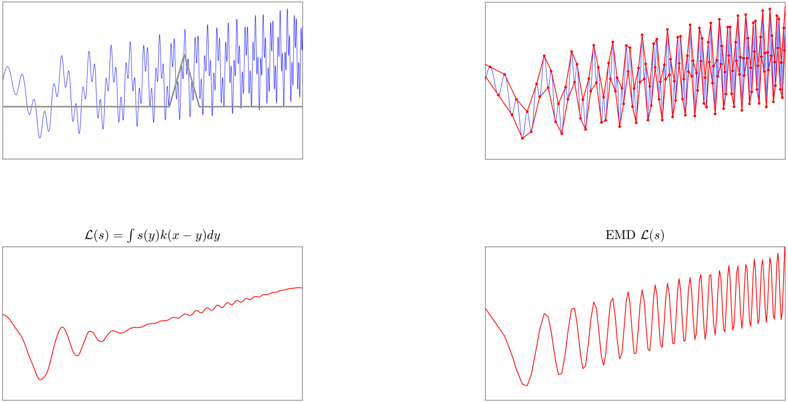
The approximation error of $\boldsymbol{IMF}_j$ with respect to the continuous algorithm modes IMF_j is proportional to $\log(1/\delta)/n$

Drawbacks



What's happening? Let's take a look at the instantaneous frequencies

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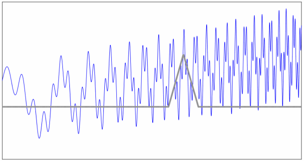

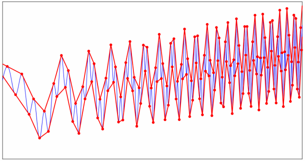

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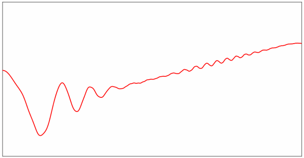
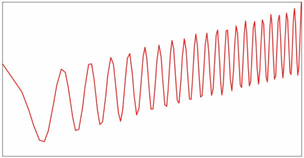


EMD $\mathcal{L}(s)$



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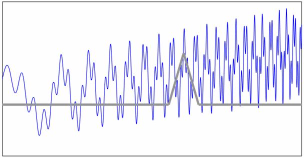

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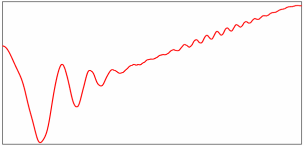
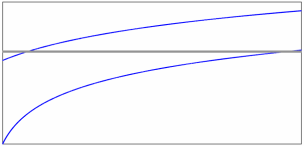


Instantaneous Frequencies



$$\widehat{\mathcal{L}(s)}(\xi) = \widehat{s}(\xi) \cdot \widehat{k}(\xi)$$

IF does not work with non-disjoint bands of frequencies

Adaptive Local Iterative Filtering

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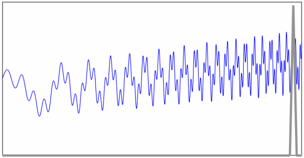

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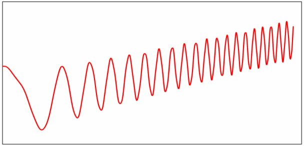

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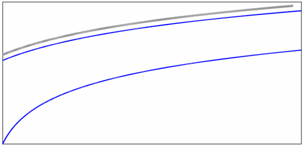
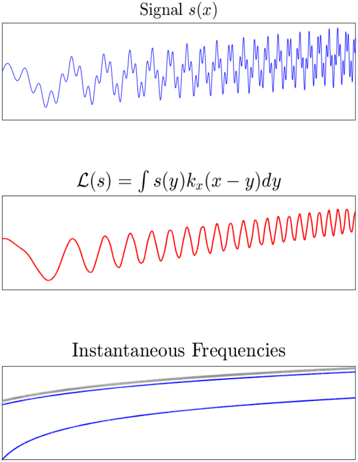


Adaptive Local Iterative Filtering



Given the signal $s(x)$, fix the filter

$$k_x(y) := k(\ell(x)^{-1}y)\ell(x)^{-1}$$

where ideally $\ell(x) \sim \xi_0/f(x)$, with $f(x)$ being the instantaneous frequency of the higher-frequency IMF.

Apply iteratively the filter through sifting

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$$IMF = IMF \cup \{\mathcal{S}^\infty(s)\}$$
$$s = s - \mathcal{S}^\infty(s)$$

ALIF is now as flexible as EMD, and empirically converges, but..

- No structure, not fast as IF ($O(n^2)$ against $O(n\log(n))$)
- Has no clean formal analysis since it is not a convolution
- $\mathcal{S}^\infty(s)$ is not always convergent (in the discrete setting) even with a stopping condition

Discrete ALIF and SALIF

$$\mathcal{S}_{ALIF}(s)(x) = s(x) - \int_0^1 s(y) k_x (x-y) dy|_{x=ah} \quad \sim \quad \boldsymbol{s}_a - \frac{1}{N} \sum_{b=1}^{N} k\left(\frac{(a-b)h}{\ell(ah)}\right) \frac{1}{\ell(ah)} \boldsymbol{s}_b$$

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- $\mathcal{S}_{ALIF}^{\infty}(\boldsymbol{s})$ converges when $|\lambda_i(I-K)| < 1 \vee \lambda_i(I-K) = 1$
- Converges to the kernel of K
- K may have sparse **[B., Cicone 2022]** negative eigenvalues, so the convergence is not always assured

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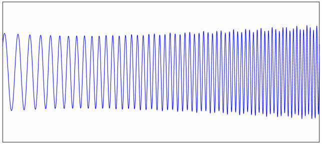
$\mathcal{S}_{SALIF}(\mathbf{s}) := \mathbf{s} - K^TK\mathbf{s} = (I-K^TK)\mathbf{s}$

- K^TK Has the same kernel of K
- $1 \geq \lambda_i(K^TK) \geq 0$ so $\mathcal{S}^{\infty}_{SALIF}(\mathbf{s})$ **always converges**
- The method is **way slower**: the cost per iteration is doubled and the eigenvalues are closer to zero, so it's harder to extract the components

$T \times 1/20$ $\qquad N = 3000$

ALIF $\mathcal{S}^p(s)$ - Finished



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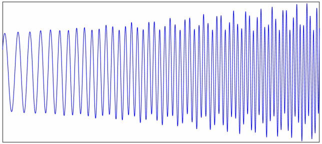

Results about SALIF

$$\mathcal{S}(\boldsymbol{s}) = (I - K^T K)\boldsymbol{s} \qquad 1 \geq \lambda_i(K^T K) \geq 0$$

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Since $\|K^T K\| \leq 1$ and it is Hermitian, we can recover some of the IF good properties:

Theorem (B. 2023)

For any vectors $\boldsymbol{h}, \boldsymbol{s}$ let K be any $n \times n$ Hermitian matrix with spectrum in $[0, 1]$. Then

$$\|\mathcal{S}^m(\boldsymbol{s} + \boldsymbol{h}) - \mathcal{S}^m(\boldsymbol{s})\| \leq \|\boldsymbol{h}\|.$$

If now the filters and m_j are fixed, for $\boldsymbol{IMF}_j$ the modes generated by $\boldsymbol{s}$ and for $\boldsymbol{IMF}_j^*$ generated by $\boldsymbol{s} + \boldsymbol{h}$, we have

$$\sum_j \|\boldsymbol{IMF}_j^* - \boldsymbol{IMF}_j\|^2 \leq \|\boldsymbol{h}\|^2.$$

Theorem (B. 2023)

Given $\delta > 0$, $\boldsymbol{s}$, then

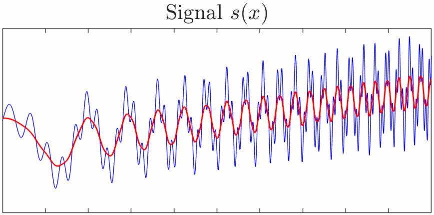
$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|\boldsymbol{s}\|} \implies \|\mathcal{S}^{m+1}(\boldsymbol{s}) - \mathcal{S}^m(\boldsymbol{s})\| < \delta$$

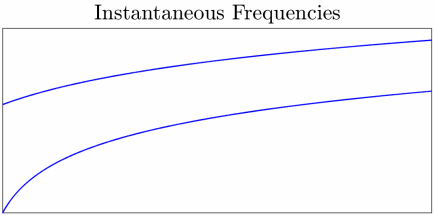
Theorem (B. 2023)

$\sum_j \|\boldsymbol{IMF}_j\|^2 \leq \|\boldsymbol{s}\|^2$. *Thus, there is a finite number of relevant IMF, i.e.* $\|\boldsymbol{IMF}_j\| > \eta$

Resampled Iterative Filtering

Resampling





Resampling Function $G(y)$

Recall that in ALIF the length $\ell(x)$ is computed as $\xi_0/f(x)$ where $f(x)$ is the highest instantaneous frequency for the IMFs of the signal $s(x)$.
From now on $\xi_0 = 1$.

Example: The Instantaneous Frequency of $s(x) = \cos(\alpha(x))$ is $\alpha'(x)$ if it is regular enough. In this case, $\ell(x) = 1/\alpha'(x)$.

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In the Resampled IF (RIF), we instead operate a IF loop to the resampled stationary signal $s(G(y))$ where

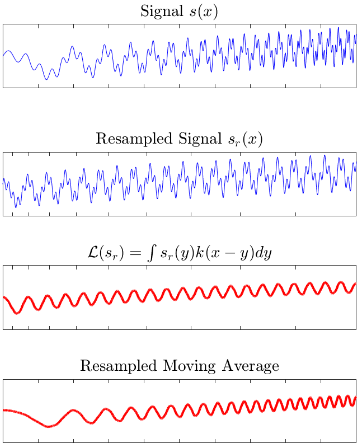
$$G^{-1}(z) = \int_0^z \frac{1}{\ell(x)} dx$$

Example: In the previous example, $G^{-1}(z) = \int_0^z \alpha'(x) = \alpha(z) - \alpha(0)$ so that

$$s(G(y)) = \cos(\alpha(G(y))) = \cos(\alpha(0) + y)$$

is a stationary signal with frequency equal to $\xi_0 = 1$

Resampled Iterative Filtering



Given the signal $s(x)$, compute the resampling

$$s_r(x) := s(G(x)) \qquad G^{-1}(z) = \int_0^z \frac{1}{\ell(x)}dx$$

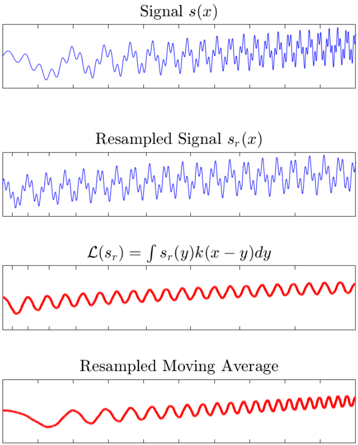
and apply iteratively the filter through convolution

$$\mathcal{S}(f) := f(x) - \int f(y)k(x-y)dy$$

$$IMF = IMF \cup \{\mathcal{S}^\infty(s_r)(G^{-1}(x))\}$$

$$s = s - \mathcal{S}^\infty(s_r)(G^{-1}(x))$$

Resampled Iterative Filtering



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We have an algorithm that is

- As flexible as ALIF and SALIF
- Efficient as Fast IF, the resampling is outside the iterations and has the same complexity as the FFT, thus way faster than ALIF and SALIF
- Differently from ALIF, $\mathcal{S}^\infty(s_r)$ is **always** convergent because it is an IF iteration. In particular, given a stopping criterion with $\delta > 0$ we have the same results that limit the number of iterations.

Theorem

Given $0 \le \widehat{k} \le 1$, $\delta > 0$, $s_r(x) \in L^2(\mathbb{R})$, *then*

$$\frac{m^m}{(m+1)^{m+1}} < \frac{\delta}{\|s_r\|}$$

implies $\|\mathcal{S}^{m+1}(s_r) - \mathcal{S}^m(s_r)\| < \delta$

Theorem

For any $h, s_r \in L^2$

$$\|\mathcal{S}^m(s_r + h) - \mathcal{S}^m(s_r)\| \le \|h\|$$

Fast Discrete RIF

$$\widehat{\mathcal{S}^m(\boldsymbol{s}_r)} = \boldsymbol{k}^{\circ m} \circ \widehat{\boldsymbol{s}}_r$$

$$\|\mathcal{S}^{m+1}(\boldsymbol{s}_r) - \mathcal{S}^m(\boldsymbol{s})_r\| < \delta \iff \|\boldsymbol{k}^{\circ m} \circ (\boldsymbol{k} - \boldsymbol{e}) \circ \widehat{\boldsymbol{s}}_r\| < \delta$$

The stopping condition is checked on $\boldsymbol{k}$ and $\widehat{\boldsymbol{s}}_r$ with linear cost + 2 DFT

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We don't know if we can still recover

- Global perturbation results
- Intrinsic relation with $\widehat{\boldsymbol{s}}$
- Limited number of meaningful IMFs

Non-Stationary Error Bounds

Let us suppose that the signal $s(x)$ is a linear combination of non-stationary components

$$s(x) := \sum_{j=1}^{M} a_j g_j(x) \qquad g_j(x) = \cos(\alpha_j(x)), \qquad |a_j| \leq P$$

$$s_r(z) := \sum_{j=1}^{M} a_j h_j(z) \qquad h_j(z) = \cos(\alpha_j(\alpha_1^{-1}(2\pi s z)) = \cos(\beta_j(z)) \qquad h_1(z) = \cos(2\pi s z)$$

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Theorem (B. 2023)

Let $\beta : \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function with* $\beta'(x) \in [a, b]$ *1-periodic,* $0 < a < b$, $R := b - a$. *Let* $f(x) := \cos(\beta(x))$ *and let* $f(x)_N$ *be the N-tail of its Fourier series, and* $G := 2\pi N - b > 0$

$$\|f(x)_N\|_2^2 \leq \min\left\{\left(\frac{b}{G + b + 2\pi}\right)^2, \frac{R^2}{\pi^3 G}\right\}$$

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If now $j > 1$, $f(z) = h_j(z)$ and $N = s - 1$, then $P\|f(x) - f(x)_N\|_2$ is a bound on the perturbation of the IMF caused by the j-th component h_j, and it is proportional to both

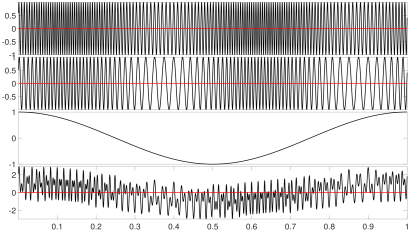
$$\frac{b}{G + b + 2\pi} = \frac{\max_z \beta_j'(z)}{2\pi s} = \max_x \frac{\alpha_j'(x)}{\alpha_1'(x)} \qquad \text{(low for far frequencies)}$$

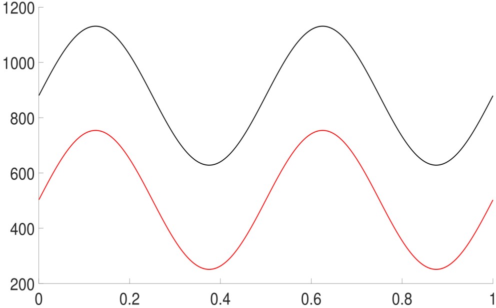
$$R = \max_z \beta_j'(z) - \min_z \beta_j'(z) = 2\pi s \left(\max_x \frac{\alpha_j'(x)}{\alpha_1'(x)} - \min_x \frac{\alpha_j'(x)}{\alpha_1'(x)} \right) \qquad \text{(zero if same shape)}$$

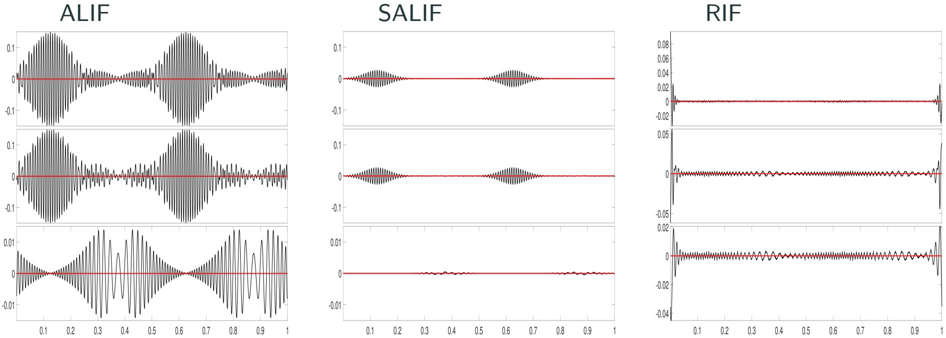
Numerical Experiments

Experiment 1

$N = 8000$
$h_1(x) = \cos(20\cos(4\pi t) - 160\pi t)$
$h_2(x) = \cos(20\cos(4\pi t) - 280\pi t)$
$h_3(x) = \cos(2\pi t)$
$h(x) = h_1(x) + h_2(x) + h_3(x)$

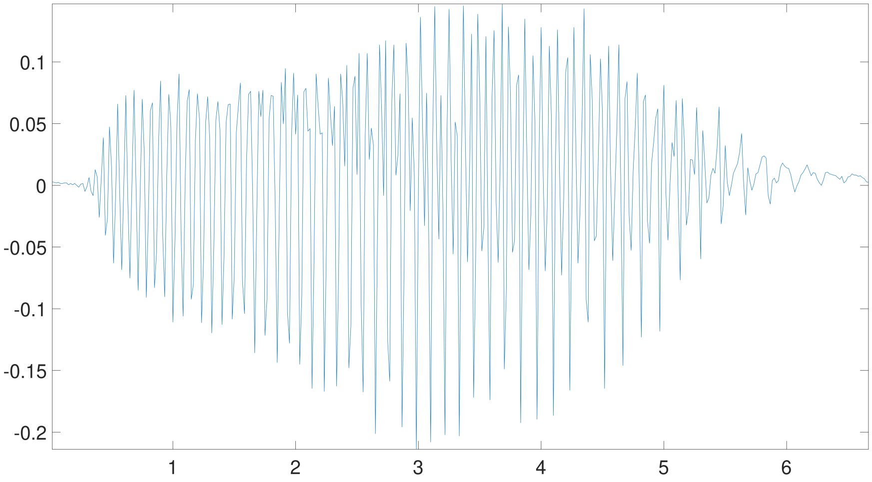






	Time	err1	err2	err3	Niter1	Niter2
ALIF	4.0860	0.070388	0.071158	0.008549	18	2
SALIF	19.7919	0.010054	0.010055	0.000161	353	5
RIF	1.4724	0.003426	0.003292	0.000908	81	11

Experiment 2



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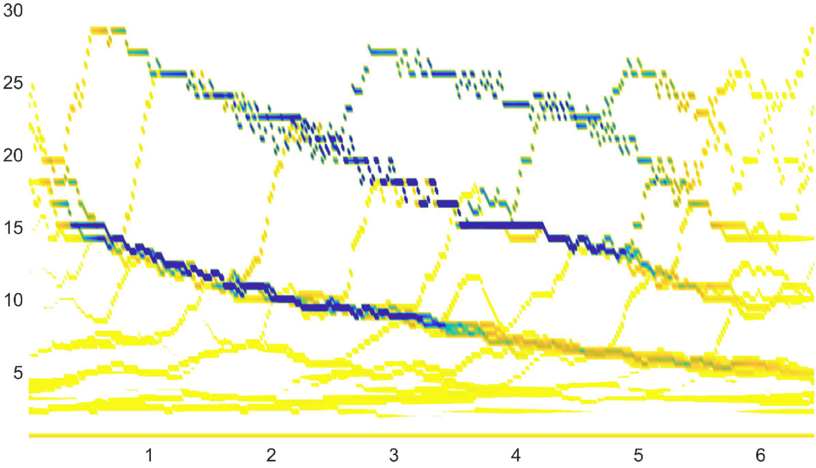
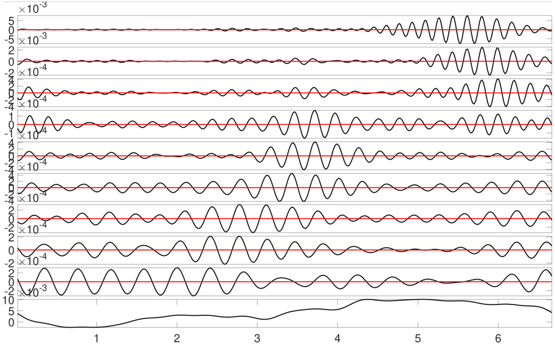
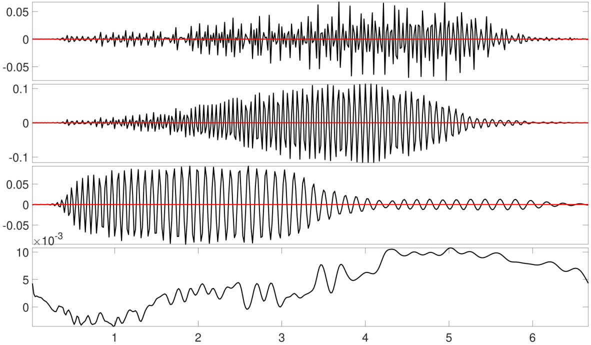


IF



RIF



Conclusions and Future Works

We developed Algorithms and Theory for

- SALIF - Stable, Flexible, Convergent but very Slow
- RIF - Flexible, Convergent, Fast but may introduce inaccuracies

Moreover RIF proves himself also Aliasing-Free and we also expanded the theory of IF.

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Moreover RIF proves himself also Aliasing-Free and we also expanded the theory of IF.

Still to do:

- Better exploit the order of zero of the filter
- Further analysis of IF for non-stationary and AM components
- We can use RIF to better study ALIF through the relation between $G(x)$ and $\ell(x)$
- Better ways to compute $G(x)$ without relying on $\ell(x)$
- Improve the error bounds, since they prove to be empirically better
- How perturbation affect the output of RIF

Thank You!

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