Spectral Measures

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Abstract
The theory of spectral symbols links sequences of matrices with measurable functions expressing their asymptotic eigenvalue distributions. Usually, a sequence admits several spectral symbols, and it is not clear if a canonical one exists. Here we present a way to connect the sequences with the space of probability measure, so that each sequence admits a uniquely determined measure. The methods used are similar to those employed in the theory of Generalized Locally Toeplitz (GLT) sequences: a goal of this present contribution is in fact that of explaining how the two concepts are connected.

1 Introduction

A *Matrix Sequence* is an ordered collection of complex valued matrices with increasing size, and is usually denoted as $\{A_n\}_n$, where $A_n \in \mathbb{C}^{n\times n}$. We will refer to the space of matrix sequences with the notation

$$\mathscr{E} := \{\{A_n\}_n : A_n \in \mathbb{C}^{n\times n}\}.$$

It is often observed in practice that matrix sequences $\{A_n\}_n$, generated by discretization methods applied to linear differential equations possess a *Spectral Symbol*, that is a measurable function describing the asymptotic distribution of the eigenvalues of A_n. We recall that a spectral symbol associated with a sequence $\{A_n\}_n$ is a measurable functions $k : D \subseteq \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{l(D)} \int_D F(k(x))dx$$

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for every continuous function $F : \mathbb{C} \to \mathbb{C}$ with compact support, where D is a measurable set with finite Lebesgue measure $l(D) > 0$ and $\lambda_i(A_n)$ are the eigenvalues of A_n. In this case we write

$$\{A_n\}_n \sim_\lambda k(x).$$

We can also consider the singular values of the matrices instead of the eigenvalues. In the same setting, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{l(D)} \int_D F(|k(x)|)dx$$

for every continuous function $F : \mathbb{R} \to \mathbb{C}$ with compact support, where $\sigma_i(A_n)$ are the singular values of A_n, then $\{A_n\}_n$ possesses a *Singular Value Symbol*, and we write

$$\{A_n\}_n \sim_\sigma k(x).$$

The space of matrix sequences is a complete pseudometric space when endowed with a pseudometric inducing the *Approximating Classes of Sequences* (acs) convergence, that we will redefine in the next section. One fundamental property of this metric is that it identifies sequences that differ by a sequence admitting zero as singular value symbol (called zero distributed sequences). In particular, it has been shown that such sequences share the same singular value symbol, but the distance between two sequences with the same singular value symbol is not usually zero.

The main observation of this note is that for any measurable function $k(x)$, the operator

$$\phi(F) := \int_D F(k(x))dx \qquad \phi : C_c(\mathbb{C}) \to \mathbb{C}$$

is linear and continuous and can be represented by an unique probability measure μ. We call μ a *Spectral Measure*, and we associate it with any sequence $\{A_n\}_n$ that has $k(x)$ as spectral symbol. It turns out that if a sequence admits a spectral measure, then it is uniquely determined, differently from the spectral symbols. The space of probability spectral measures is moreover a complete metric space with the Lévy-Prokhorov distance π, and it correspond to a pseudometric d' on matrices called *Modified Optimal Matching* distance. The main result is that d' identifies sequences admitting the same spectral symbol, differently from the acs distance.

Theorem 1. *If* $\{A_n\}_n \sim_\lambda f(x)$*, then*

$$\{B_n\}_n \sim_\lambda f(x) \iff d'(\{A_n\}_n, \{B_n\}_n) = 0.$$

A different approach to the uniqueness problem for the spectral symbol is embodied in the theory of GLT sequences. For specific sequences, called *Generalized Locally Toeplitz* (GLT) sequences, we can choose one of their symbols, and denote it as *GLT Symbol* of the sequence

$$\{A_n\}_n \sim_{GLT} k(x, \theta).$$

In the case of diagonal matrix sequences, the choice of one symbol can be seen as a particular sorting of their eigenvalues, as expressed in the following theorem, proved in the last section, and which represents a generalization of the results in [3].

Theorem 2. *Given a diagonal sequence $\{D_n\}_n$ and one of its spectral symbols $k : [0,1] \to \mathbb{C}$, then*

$$\{P_n D_n P_n^T\} \sim_{GLT} k(x) \otimes 1$$

for some P_n permutation matrices.

The paper is organized in the following way: In Section 2 we recall basic definitions like the acs convergence, the optimal matching distance d and the theory of GLT sequences. Moreover, we define the modified optimal matching distance d' since it is a slight variation of d, and we discuss how it is connected to d_{acs}. In Section 3 we introduce the spectral measures and we study their relationships with the spectral symbols. In particular, we notice how the vague convergence and the Lévy-Prokhorov distance π on the probability measures lead to a reformulation of the definition of spectral symbol/measure. In Section 4, we prove that the pseudometrics π and d' are actually equivalent, and we explain how this fact leads to the proofs of the above reported theorems.

2 Prerequisites

2.1 Complete Pseudometrics

The space of matrix sequences that admit a spectral symbol on a fixed domain D has been shown to be closed with respect to a notion of convergence called the Approximating Classes of Sequences (acs) convergence. This notion and this result are due to Serra-Capizzano [11], but were actually inspired by Tilli's pioneering paper on LT sequences [12]. Given a sequence of matrix sequences $\{B_{n,m}\}_{n,m}$, it is said to be acs convergent to $\{A_n\}_n$ if there exists a sequence $\{N_{n,m}\}_{n,m}$ of "small norm" matrices and a sequence $\{R_{n,m}\}_{n,m}$ of "small rank" matrices such that for every m there exists n_m with

$$A_n = B_{n,m} + N_{n,m} + R_{n,m}, \qquad \|N_{n,m}\| \le \omega(m), \qquad \mathrm{rk}(R_{n,m}) \le nc(m)$$

for every $n > n_m$, and

$$\omega(m) \xrightarrow{m\to\infty} 0, \qquad c(m) \xrightarrow{m\to\infty} 0.$$

In this case, we will use the notation $\{B_{n,m}\}_{n,m} \xrightarrow{acs} \{A_n\}_n$.

This notion of convergence has been shown to be metrizable on the whole space $\mathscr{E}$. Given a matrix $A \in \mathbb{C}^{n\times n}$, we can define the function

$$p(A) := \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + \sigma_i(A) \right\},$$

where $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$ are the singular values of A, and by convention $\sigma_{n+1}(A) = 0$. The function $p(A)$ is subadditive, so we can introduce the pseudometric d_{acs} on the space of matrix sequences

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} p(A_n - B_n).$$

It has been proved ([6],[8]) that this distance induces the acs convergence already introduced. In other words,

$$d_{acs}(\{A_n\}_n, \{B_{n,m}\}_{n,m}) \xrightarrow{m\to\infty} 0 \iff \{B_{n,m}\}_{n,m} \xrightarrow{acs} \{A_n\}_n.$$

One fundamental property of this metric is that it identifies sequences whose difference admits zero as singular value symbol (called zero distributed sequence). In particular, it has been shown that such sequences share the same singular value symbol, in case one of them admits singular value symbol.

Lemma 1. *Let $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$. We have*

$$\{A_n - B_n\}_n \sim_\sigma 0 \iff d_{acs}(\{A_n\}_n, \{B_n\}_n) = 0.$$

In this case, if $k : D \subseteq \mathbb{R}^n \to \mathbb{C}$ where D is a measurable set with finite Lebesgue measure $l(D) > 0$, then

$$\{A_n\}_n \sim_\sigma k(x) \iff \{B_n\}_n \sim_\sigma k(x).$$

In [2], has been first proved that the pseudometric d_{acs} on the space of matrix sequences is complete. In Theorem 2.2 of [4], we find sufficient conditions for a pseudometric on $\mathscr{E}$ to be complete. Here we need a different result, but the proof is almost identical.

Lemma 2. *Let d_n be pseudometrics on the space of matrices $\mathbb{C}^{n\times n}$ bounded by the same constant $L > 0$ for every n. Then the function*

$$d(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} d_n(A_n, B_n)$$

is a complete pseudometric on the space of matrix sequences.

2.2 Optimal Matching Distance

Let $v, w \in \mathbb{C}^n$ be vectors with components

$$v = [v_1, v_2, \dots, v_n], \quad w = [w_1, w_2, \dots, w_n].$$

We recall the pseudometric on $\mathbb{C}^n$ called *Optimal Matching Distance* defined in Bhatia's book [5].

Definition 1. Given $v, w \in \mathbb{C}^n$, the pseudometric of the optimal matching distance is defined as

$$d(v,w) := \min_{\sigma \in S_n} \max_{i=1,\dots,n} |v_i - w_{\sigma(i)}|,$$

where S_n is the symmetric group of permutation of n objects.

Given $A \in \mathbb{C}^{n\times n}$, let $\Lambda(A) \in \mathbb{C}^n$ be the vector of the eigenvalues. We can extend the distance d to matrices in the following way.

Definition 2. Given $A, B \in \mathbb{C}^{n\times n}$, we define

$$d(A,B) := d(\Lambda(A), \Lambda(B)).$$

Notice that the order of the eigenvalues in $\Lambda(A)$ and $\Lambda(B)$ does not affect the quantity $d(A,B)$. It is easy to see that d is still a pseudometric on $\mathbb{C}^{n\times n}$. This is still not enough for our purposes, since we want a distance that sees two matrices differing for few eigenvalues as very similar. For this reason, we modify the previous metric, and we introduce a new function d' called *Modified Optimal Matching Distance*.

Definition 3. Given $v, w \in \mathbb{C}^n$, the modified optimal matching distance is defined as

$$d'(v,w) := \min_{\sigma \in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v - w_\sigma|_i^{\downarrow} \right\},$$

where

$$|v - w_\sigma| = [|v_1 - w_{\sigma(1)}|, |v_2 - w_{\sigma(2)}|, \dots, |v_n - w_{\sigma(n)}|]$$

and $|v - w_\sigma|_i^{\downarrow}$ is the i-th greatest element in $|v - w_\sigma|$, with the convention $|v - w_\sigma|_{n+1}^{\downarrow} := 0$.

Given $A, B \in \mathbb{C}^{n\times n}$, we define

$$d'(A,B) := d'(\Lambda(A), \Lambda(B))$$

and if $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, we can also define

$$d'(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} d'(A_n, B_n).$$

Notice that $d'(v,w) \le 1$ for every $v, w \in \mathbb{C}^n$, so $d'(A,B) \le 1$ for every pair of matrices of the same size, and $d'(\{A_n\}_n, \{B_n\}_n) \le 1$ for every pair of sequences $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$. We referred to d' as a distance, but we need to prove it.

Lemma 3. *The function d' is a complete pseudometric on $\mathscr{E}$.*

Proof. Let us prove that d' is a pseudometric on $\mathbb{C}^n$. First, it is easy to see that $d'(v,w)$ is always a finite nonnegative real number, and it is symmetric since

$$\begin{aligned} d'(v,w) &= \min_{\sigma\in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v - w_\sigma|_i^{\downarrow} \right\} \\ &= \min_{\sigma\in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |w - v_{\sigma^{-1}}|_i^{\downarrow} \right\} = d'(w,v). \end{aligned}$$

Moreover, given any $\tau \in S_n$, we have

$$\begin{aligned} d'(v,w) &= \min_{\sigma\in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v - w_\sigma|_i^{\downarrow} \right\} \\ &= \min_{\sigma\in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v_\tau - w_{\sigma\tau}|_i^{\downarrow} \right\} = d'(v_\tau,w), \end{aligned}$$

so we can permute the elements of the vectors as we like. Let $v,w,z \in \mathbb{C}^n$ and let us sort their elements in such a way that

$$\begin{aligned} d'(v,w) &= \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v_i - w_i| \right\}, \\ d'(w,z) &= \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |w - z|_i^{\downarrow} \right\}, \end{aligned}$$

meaning that the permutation realizing the minimum in both cases is the identity, and that $|v_i - w_i| \ge |v_j - w_j|$ whenever $i \le j$. Moreover, let s,r,q be the greatest indices that satisfy

$$d'(v,w) = \frac{s-1}{n} + |v_s - w_s|, \qquad d'(w,z) = \frac{r-1}{n} + |w_q - z_q|.$$

Let I,J be two sets of indices defined as

$$I = \{1,2,\dots,s-1\}, \qquad J = \{j : |w_j - z_j| > |w_q - z_q|\}.$$

Notice that $\#I = s-1$ and $\#J = r-1$. Let us consider two cases.

- Suppose $I \cup J = \{1,\dots,n\}$. We obtain that

  $$\#I + \#J = r + s - 2 \ge n$$

  and hence

  $$d'(v,z) \le 1 \le \frac{s-1}{n} + \frac{r-1}{n} \le d'(v,w) + d'(w,z).$$

- Suppose $I \cup J \ne \{1,\dots,n\}$. Let k be the index not belonging to $I \cup J$ that maximizes $|v_i - z_i|$. If we consider the identity permutation, we deduce that

  $$d'(v,z) \le \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v - z|_i^{\downarrow} \right\},$$

but the number of indices such that $|v_i - z_i|$ is greater than $|v_k - z_k|$ is at most $\#I \cup J \le r+s-2$, and consequently

$$d'(v,z) \le \frac{r+s-2}{n} + |v_k - z_k|.$$

The index k does not belong to I or to J, so

$$|v_k - w_k| \le |v_s - w_s|, \qquad |w_k - z_k| \le |w_q - z_q|.$$

From the latter we infer that

$$\begin{aligned} d'(v,z) &\le \frac{r+s-2}{n} + |v_k - z_k| \\ &\le \frac{s-1}{n} + |v_k - w_k| + \frac{r-1}{n} + |w_k - z_k| \\ &\le \frac{s-1}{n} + |v_s - w_s| + \frac{r-1}{n} + |w_q - z_q| \\ &= d'(v,w) + d'(w,z). \end{aligned}$$

This shows that d' is a pseudometric on $\mathbb{C}^n$ and consequently it is a pseudometric even on $\mathbb{C}^{n\times n}$. Thanks to Lemma 2, we can conclude that d' is a complete pseudometric on $\mathscr{E}$. ☐

In the general case, the two pseudometrics have no common features, but, when dealing with diagonal matrices, we can prove the following lemma.

Lemma 4. *Given $\{D_n\}_n, \{D'_n\}_n \in \mathscr{E}$ sequences of diagonal matrices, there exists a sequence $\{P_n\}_n$ of permutation matrices such that*

$$d'(\{D'_n\}_n, \{D_n\}_n) = d_{acs}(\{D'_n\}_n, \{P_n D_n P_n^T\}_n).$$

Proof. Let v^n and v'^n be the vectors of the ordered diagonal entries of D_n and D'_n, so that

$$v^n_i := [D_n]_{i,i}, \qquad v'^n_i := [D'_n]_{i,i}.$$

Let $\tau_n \in S_n$ be the permutations satisfying

$$\begin{aligned} d'(D'_n, D_n) &= \min_{\sigma \in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v'^n - v^n_\sigma|^\downarrow_i \right\} \\ &= \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v'^n - v^n_{\tau_n}|^\downarrow_i \right\}. \end{aligned}$$

Let also P_n be the permutation matrices associated to τ_n. We know that

$$\begin{aligned} p(D'_n - P_n D_n P_n^T) &= \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + \sigma_i(D'_n - P_n D_n P_n^T) \right\} \\ &= \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v'^n - v^n_{\tau_n}|^{\downarrow}_i \right\} \\ &= d'(D'_n, D_n). \end{aligned}$$

As a consequence

$$\begin{aligned} d_{acs}(\{D'_n\}_n, \{P_n D_n P_n^T\}_n) &= \limsup_{n\to\infty} p(D'_n - P_n D_n P_n^T) \\ = \limsup_{n\to\infty} d'(D'_n, D_n) &= d'(\{D'_n\}_n, \{D_n\}_n). \end{aligned}$$

□

2.3 GLT Matrix Sequences

A matrix sequence $\{A_n\}_n$ may have several different singular value symbols, even on the same domain. For specific sequences, called *Generalized Locally Toeplitz* (GLT) sequences, we can choose one of their symbols, and denote it as *GLT Symbol* of the sequence

$$\{A_n\}_n \sim_{GLT} k(x,\theta).$$

where the chosen symbols have all the same domain $D = [0,1] \times [-\pi, \pi]$. If we denote with $\mathscr{M}_D$ the set of measurable functions on D, and with $\mathscr{G}$ the set of GLT sequences, then the choice of the symbol can be seen as a map

$$S : \mathscr{G} \to \mathscr{M}_D.$$

Both $\mathscr{G}$ and $\mathscr{M}_D$ are $\mathbb{C}$ algebras and pseudometric spaces with the distances d_{acs} and d_m, inducing respectively the acs convergence and the convergence in measure. In [9] and in [2] several properties of the map S are proved.

Theorem 3.

1. *S is an homomorphism of $\mathbb{C}$ algebras. Given $\{A_n\}_n, \{B_n\}_n \in \mathscr{G}$ and $c \in \mathbb{C}$, we have that*

$$S(\{A_n + B_n\}_n) = S(\{A_n\}_n) + S(\{B_n\}_n),$$
$$S(\{A_n B_n\}_n) = S(\{A_n\}_n) \cdot S(\{B_n\}_n),$$
$$S(\{cA_n\}_n) = cS(\{A_n\}_n).$$

2. *The kernel of S are exactly the zero-distributed sequences.*
3. *S preserves the distances. Given $\{A_n\}_n, \{B_n\}_n \in \mathscr{G}$ we have*

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = d_m(S(\{A_n\}_n), S(\{B_n\}_n)).$$

4. *S is onto. All measurable functions are GLT symbols.*
5. *GLT symbols are singular value symbols:*

$$\{A_n\}_n \in \mathscr{G} \implies \{A_n\}_n \sim_\sigma S(\{A_n\}_n)$$

6. *The graph of S is closed in $\mathscr{G} \times \mathscr{M}_D$. If $\{B_{n,m}\}_{n,m}$ are sequences in $\mathscr{G}$ that converge acs to $\{A_n\}_n$, and their symbols converge in measure to $k(x,\theta)$, then $S(\{A_n\}_n) = k(x,\theta)$.*

The diagonal sampling sequences are denoted as $\{D_n(a)\}_n$, where $a : [0,1] \to \mathbb{C}$ is a measurable function, and

$$D_n(a) = \operatorname*{diag}_{i=1,\dots,n} a\left(\frac{i}{n}\right) = \begin{pmatrix} a\left(\frac{1}{n}\right) & & & \\ & a\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & a(1) \end{pmatrix}$$

It is easy to verify that when $a : [0,1] \to \mathbb{C}$ is an almost everywhere (a.e.) continuous function, we have $\{D_n(a)\}_n \sim_{\sigma,\lambda} a(x)$. Furthermore, if $a(x)$ is continuous, we know that these sequences have as GLT symbol

$$\{D_n(a)\}_n \sim_{GLT} a(x) \otimes 1,$$

where $a \otimes 1 : [0,1] \times [-\pi,\pi] \to \mathbb{C}$ is a function constant in the second variable. This is not true for every $a(x)$ measurable, so we resort to the following result.

Lemma 5. *Given any $a : [0,1] \to \mathbb{C}$ measurable function, and $a_m \in C([0,1])$ continuous functions that converge in measure to $a(x)$, there exists an increasing and unbounded map $m(n)$ such that*

$$\{D_n(a_{m(n)})\}_n \sim_{GLT} a(x) \otimes 1 \qquad \{D_n(a_{m(n)})\}_n \sim_\lambda a(x)$$

Proof. Easy corollary of Lemma 3.4 and Theorem 3.1 in [3]. □

3 Spectral Measures

3.1 Radon measures

Let $\{A_n\}_n \in \mathscr{E}$ be a sequence with a spectral symbol $k(x)$ with domain D. By definition, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G(\lambda_i(A_n)) = \frac{1}{l(D)} \int_D G(k(x))dx.$$

Let $\phi : C_c(\mathbb{C}) \to \mathbb{C}$ be the functional defined as

$$\phi(G) := \frac{1}{l(D)} \int_D G(k(x))dx.$$

The latter is a continuous and linear map, and if we restrict it to real valued compacted supported functions, it is also a positive operator, since

$$G(x) \geq 0 \quad \forall x \in \mathbb{C} \implies \phi(G) = \frac{1}{l(D)} \int_D G(k(x))dx \geq 0.$$

Let us now recall Riesz Theorem ([1]).

Theorem 4 (Riesz). *Let $\phi : C_c(X) \to \mathbb{R}$ be a positive linear and continuous function, where X is an Hausdorff and locally compact space. There exists an uniquely determined Radon positive measure μ such that*

$$\phi(F) = \int_X F d\mu \qquad \forall F \in C_c(X).$$

If $G \in C_c(\mathbb{C})$ is a complex valued map, we can always decompose it into $G = G_1 + iG_2$ where G_1 and G_2 are real valued and supported on a compact. Since ϕ is linear, we get

$$\phi(G) = \phi(G_1) + i\phi(G_2) = \int_{\mathbb{C}} G_1 d\mu + i \int_{\mathbb{C}} G_2 d\mu = \int_{\mathbb{C}} G d\mu$$

so ϕ induces an unique measure μ. We can thus define a *Spectral Measure*.

Definition 4. Given $\{A_n\}_n \in \mathscr{E}$, we say that it has a spectral measure μ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G(\lambda_i(A_n)) = \int_{\mathbb{C}} G d\mu$$

for every $G \in C_c(\mathbb{C})$.

Let $G_m \in C_c(\mathbb{C})$ be a sequence of nonnegative real valued maps such that $\|G_m\|_\infty \leq 1$ and

$$G_m(x) = 1 \qquad \forall\, |x| \leq m.$$

We find that

$$\int_{\mathbb{C}} G_m d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G_m(\lambda_i(A_n)) \leq 1$$

and hence

$$\mu(\mathbb{C}) = \lim_{m\to\infty} \mu\left(\{x : |x| \leq m\}\right) \leq \limsup_{m\to\infty} \int_{\mathbb{C}} G_m d\mu \leq 1.$$

This proves that all the measures we consider are finite. Since all the finite measures over the Borelian set are Radon, we will now simply say "measure" instead of "Radon measure". We showed that any measurable function induces a finite measure, but we can actually prove that it induces a probability measure, and also that any probability measure is induced by a function.

Lemma 6. *Let $D \subseteq \mathbb{R}^n$ be a measurable set with finite non zero measure. Then, for any $k \in \mathscr{M}_D$ there exists a probability measure μ such that*

$$\frac{1}{l(D)}\int_D G(k(x))dx = \int_{\mathbb{C}} G d\mu \qquad \forall G \in C_c(\mathbb{C}).$$

Let J be the real interval $[0,1]$. Then for every probability measure μ there exists a measurable function $k \in \mathscr{M}_J$ such that

$$\int_0^1 G(k(x))dx = \int_{\mathbb{C}} G d\mu \qquad \forall G \in C_c(\mathbb{C}).$$

Proof.
Given $k \in \mathscr{M}_D$, we already showed that Riesz Theorem identifies an unique finite measure μ such that

$$\frac{1}{l(D)}\int_D G(k(x))dx = \int_{\mathbb{C}} G d\mu \qquad \forall G \in C_c(\mathbb{C}).$$

Let us consider $M > 0$ and denote

$$\chi_M(x) = \begin{cases} 1 & |x| \le M, \\ 0 & |x| > M. \end{cases}$$

Moreover, let us fix $\varepsilon > 0$, so that for every $M > 0$ we can find $G_M \in C_c(\mathbb{C})$ such that

$$\chi_M(x) \le G_M(x) \le \chi_{M+\varepsilon}(x) \qquad \forall x \in \mathbb{C}.$$

We infer

$$\int_{\mathbb{C}} \chi_{M-\varepsilon} d\mu \le \int_{\mathbb{C}} G_{M-\varepsilon} d\mu = \frac{1}{l(D)}\int_D G_{M-\varepsilon}(k(x))dx \le \frac{1}{l(D)}\int_D \chi_M(k(x))dx,$$
$$\frac{1}{l(D)}\int_D \chi_M(k(x))dx \le \frac{1}{l(D)}\int_D G_M(k(x))dx = \int_{\mathbb{C}} G_M d\mu \le \int_{\mathbb{C}} \chi_{M+\varepsilon} d\mu$$

so that

$$\int_{\mathbb{C}} \chi_{M-\varepsilon} d\mu \le \frac{1}{l(D)}\int_D \chi_M(k(x))dx \le \int_{\mathbb{C}} \chi_{M+\varepsilon} d\mu.$$

When we let ε go to zero, we obtain that the integrals coincide on the indicator functions of closed intervals

$$\int_{\mathbb{C}} \chi_M d\mu = \frac{1}{l(D)}\int_D \chi_M(k(x))dx.$$

The symbol $k(x)$ is a measurable function, so it is *Sparsely Unbounded*, meaning that

$$\lim_{M\to\infty} l(\{x : |k(x)| > M\}) = \lim_{M\to\infty}\int_D \chi_{|x|>M}(k(x))dx = 0.$$

With the latter, we can conclude that μ is a probability measure

$$\mu(\mathbb{C}) = \lim_{M\to+\infty}\int_{\mathbb{C}} \chi_{|x|\le M} d\mu = \lim_{M\to\infty} \frac{1}{l(D)}\int_D \chi_{|x|\le M}(k(x))dx = 1.$$

Given any probability measure μ, we know that the space $(\mathbb{C},\mu)$ is a *Standard Probability Space*, meaning that it is isomorphic to a space $X = I \sqcup E$, where I is a real finite interval with the Lebesgue measure, and $E = \{x_1, x_2, \dots\}$ is a discrete numerable set with an atomic measure ν. In particular, the isomorphism $\varphi : \mathbb{C} \to X$ satisfies

$$\mu(U) = l \oplus \nu(\varphi(U)) \qquad \forall U \in \mathscr{B}(\mathbb{C}).$$

and if the atomic measure is $\nu = \sum_{i=1}^{+\infty} c_i\delta_{x_i}$, then

$$1 = \mu(\mathbb{C}) = l \oplus \nu(X) = l(I) + \sum_{i=1}^{+\infty} c_i.$$

If we call $S = \nu(X) = \sum_{i=1}^{+\infty} c_i$, then we can take $I = [S,1]$. Let $g : [0,1] \to X$ be a map defined as

$$g(x) := \begin{cases} x_k & \sum_{i=1}^{k-1} c_i \le x < \sum_{i=1}^{k} c_i, \\ x & x \ge S. \end{cases}$$

This has the same distribution as $l \oplus \nu$, since for every measurable map $G : X \to \mathbb{C}$ we obtain

$$\int_X G d(l\oplus\nu) = \sum_{i=1}^{+\infty} c_i G(x_i) + \int_S^1 G(x)dx = \int_0^1 G(g(x))dx.$$

Let now $k := \varphi^{-1} \circ g : [0,1] \to \mathbb{C}$ be a measurable function, and $G \in C_c(\mathbb{C})$. We conclude that

$$\int_{\mathbb{C}} G d\mu = \int_X G\circ\varphi^{-1} d(l\oplus\nu) = \int_0^1 G(\varphi^{-1}(g(x)))dx = \int_0^1 G(k(x))dx.$$

□

A corollary of the latter result is that any sequence with a spectral symbol admits a probability spectral measure, and also the opposite holds. Moreover, if we call $\mathbb{P}$ the set of probability measures on $\mathbb{C}$, then we can also prove that any measure $\mu \in \mathbb{P}$ is a spectral measure.

Corollary 1. *All measures in $\mathbb{P}$ are spectral measures.*

Proof. Let J be the real interval $[0,1]$. Given any $k \in \mathscr{M}_J$, then there exists a sequence of continuous functions $k_m \in \mathscr{M}_J$ converging to k in measure. Using Lemma 5, we find that k is a spectral symbol, so every function in $\mathscr{M}_J$ is a spectral symbol.

Given now a measure $\mu \in \mathbb{P}$, Lemma 6 shows that it is induced by a measurable function in $\mathscr{M}_J$, so μ is also a spectral symbol. This implies that every measure in $\mathbb{P}$ is a spectral measure. □

3.2 Vague Convergence

We notice that every matrix A_n can be associated to an atomic probability measure μ_{A_n} with support on its eigenvalues

$$\mu_{A_n} := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(A_n)}.$$

Let us return again to the definition of spectral measure and notice that it can be rewritten as

$$\lim_{n\to\infty} \int_{\mathbb{C}} G d\mu_{A_n} = \int_{\mathbb{C}} G d\mu \qquad \forall G \in C_c(\mathbb{C}).$$

This is actually the definition of *Vague Convergence* for measures.

The space $\mathbb{P}$ endowed with the vague convergence is a complete metric space, using the *Lévy-Prokhorov Metric* ([10])

$$\pi(\mu,\nu) = \inf\{\, \varepsilon > 0 \mid \mu(A) \le \nu(A^\varepsilon) + \varepsilon,\ \nu(A) \le \mu(A^\varepsilon) + \varepsilon\ \forall A \in \mathscr{B}(\mathbb{C})\,\}$$

where

$$A^\varepsilon := \{\, x \in \mathbb{C} \mid dist(x,A) < \varepsilon \,\} = \{\, x + y \mid x \in A, |y| < \varepsilon \,\}.$$

Since every matrix is associated to an atomic probability measure, we can extend the definition of π to matrices and sequences.

Definition 5. Let $A,B \in \mathbb{C}^{n\times n}$ and let μ_A, μ_B be the probability atomic measures associated to their spectra, defined as

$$\mu_A := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(A)}, \qquad \mu_B := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(B)}.$$

The Lévy-Prokhorov metric on $\mathbb{C}^{n\times n}$ is defined as

$$\pi(A,B) := \pi(\mu_A, \mu_B).$$

The Lévy-Prokhorov metric on $\mathscr{E}$ is defined as

$$\pi(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} \pi(\mu_{A_n}, \mu_{B_n}).$$

Again, we need to prove that the latter is actually a pseudometric.

Lemma 7. *The Lévy-Prokhorov metric is a pseudometric on $\mathbb{C}^{n\times n}$ and a complete pseudometric on $\mathscr{E}$.*

Proof. The Lévy-Prokhorov metric is an actual metric on the space of probability measures, so all the properties can be transferred to the space of matrices $\mathbb{C}^{n\times n}$, except for the identity of matrices with zero distance, since two different matrices may have the same eigenvalues. Thus it is a pseudometric on $\mathbb{C}^{n\times n}$, and by Lemma 2, it is a complete pseudometric on $\mathscr{E}$. □

Since every matrix is associated to an atomic probability measure, we can also use the same notation for mixed elements, like

$$\pi(A,\nu) := \pi(\mu_A,\nu).$$

The considered notation is useful since the definition of spectral measure is given by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} G(\lambda_i(A_n)) = \int_{\mathbb{C}} G d\mu \qquad \forall G \in C_c(\mathbb{C})$$

and, when $\mu \in \mathbb{P}$, it can be rewritten as

$$\{A_n\}_n \sim_\lambda \mu \iff \pi(A_n,\mu) \xrightarrow{n\to+\infty} 0.$$

The distance π on $\mathscr{E}$ is consistent with the distance between their spectral probability measures, as shown in the following result.

Lemma 8. *If $\{A_n\}_n \sim_\lambda \mu$ and $\{B_n\}_n \sim_\lambda \nu$, with $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$ and $\mu,\nu \in \mathbb{P}$, then*

$$\pi(\{A_n\}_n,\{B_n\}_n) = \pi(\mu,\nu) = \lim_{n\to\infty} \pi(A_n,B_n).$$

Proof. Using the triangular property, we infer

$$\pi(\mu,\nu) \le \pi(\mu,A_n) + \pi(A_n,B_n) + \pi(B_n,\nu),$$

$$\pi(\mu,\nu) \ge -\pi(\mu,A_n) + \pi(A_n,B_n) - \pi(B_n,\nu).$$

Thus we obtain

$$\pi(\mu,\nu) \le \liminf_{n\to\infty} \pi(\mu,A_n) + \pi(A_n,B_n) + \pi(B_n,\nu) = \liminf_{n\to\infty} \pi(A_n,B_n),$$

$$\pi(\mu,\nu) \ge \limsup_{n\to\infty} -\pi(\mu,A_n) + \pi(A_n,B_n) - \pi(B_n,\nu) = \limsup_{n\to\infty} \pi(A_n,B_n).$$

By exploiting the latter relationships we conclude that

$$\begin{gathered}\pi(\{A_n\}_n,\{B_n\}_n) = \limsup_{n\to\infty} \pi(A_n,B_n) \\ \le \pi(\mu,\nu) \le \\ \liminf_{n\to\infty} \pi(A_n,B_n) \le \pi(\{A_n\}_n,\{B_n\}_n).\end{gathered}$$

□

It is noteworthy to stress the importance of the probability condition on the measures. In fact, it is possible to find a sequence that admits a spectral measure but does not admit a spectral symbol, when the spectral measure is not a probability measure. Moreover, the Lévy-Prokhorov metric is defined only on probability measures and if $\mu_n \in \mathbb{P}$ vaguely converge to a measure not in $\mathbb{P}$, then the sequence μ_n is not even a Cauchy sequence for π.

4 Main Results

4.1 Connection between Measures

First of all, we prove that π and d' are equivalent pseudometrics on $\mathscr{E}$.

Lemma 9. *If* $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, *then*

$$\pi(\{A_n\}_n, \{B_n\}_n) \le d'(\{A_n\}_n, \{B_n\}_n) \le 2\pi(\{A_n\}_n, \{B_n\}_n).$$

Proof. Let us first prove that for any $A, B \in \mathbb{C}^{n\times n}$, we have

$$\pi(A,B) \le d'(A,B) \le 2\pi(A,B).$$

Let $\Lambda(A)$ and $\Lambda(B)$ be ordered so that

$$i < j \implies |\lambda_i(A) - \lambda_i(B)| \ge |\lambda_j(A) - \lambda_j(B)|$$

and

$$s := d'(A,B) = \frac{k-1}{n} + |\lambda_k(A) - \lambda_k(B)|.$$

In particular, we deduce that

$$|\lambda_i(A) - \lambda_i(B)| \le s \qquad \forall i \ge k$$

and consequently, for any subset $U \subseteq \mathbb{C}$, we obtain the inequality

$$\#\{\lambda_i(A) \in U,\ i \ge k\} \le \#\{\lambda_i(B) \in U^s,\ i \ge k\}.$$

Denote with μ_A and μ_B the atomic probability measures associated with A, B. Let $U \in \mathscr{B}(\mathbb{C})$ be any Borelian set and denote the cardinality of the intersection with a n-uple v as

$$Q_U(v) := \#\{i : v_i \in U\}.$$

Formally, $Q_U(v)$ is the number of elements of v inside v, counted with multiplicity. We know that

$$\begin{aligned}
\mu_A(U) &= \frac{Q_U(\Lambda(A))}{n} \\
&= \frac{Q_U(\{\lambda_i(A) : i \ge k\})}{n} + \frac{Q_U(\{\lambda_i(A) : i < k\})}{n} \\
&\le \frac{Q_{U^s}(\{\lambda_i(B) : i \ge k\})}{n} + \frac{k-1}{n} \\
&\le \frac{Q_{U^s}(\Lambda(B))}{n} + s = \mu_B(U^s) + s.
\end{aligned}$$

We symmetrically obtain also the following relation

$$\mu_B(U) \le \mu_A(U^s) + s.$$

As a consequence

$$\pi(A,B) = \inf\{\, \varepsilon > 0 \mid \mu_A(U) \le \mu_B(U^\varepsilon) + \varepsilon,\ \mu_B(U) \le \mu_A(U^\varepsilon) + \varepsilon\ \forall U \in \mathscr{B}(\mathbb{C})\,\}$$
$$\implies \pi(A,B) \le s = d'(A,B).$$

Denote now $r = \pi(A,B)$ and let T be any sub-uple of $\Lambda(A)$. If we see T as a set, then it is a finite subset of $\mathbb{C}$, so it is a Borelian set. Given any $\varepsilon > 0$ we know that

$$\mu_A(T) = \frac{Q_T(\Lambda(A))}{n} \le \mu_B(T^{r+\varepsilon}) + r + \varepsilon = \frac{Q_{T^{r+\varepsilon}}(\Lambda(B))}{n} + r + \varepsilon$$

so we deduce that

$$\frac{Q_T(\Lambda(A))}{n} \le \frac{Q_{T^r}(\Lambda(B))}{n} + r \implies Q_T(\Lambda(A)) \le Q_{T^r}(\Lambda(B)) + rn.$$

By using the fact that the map Q is integer valued, we conclude that

$$Q_T(\Lambda(A)) \le Q_{T^r}(\Lambda(B)) + \lfloor rn \rfloor.$$

The quantity $Q_T(\Lambda(A))$ is actually the cardinality of T seen as a sub-uple of $\Lambda(A)$, so for every subset T of k eigenvalues in A, even repeated, there are at least $k - \lfloor rn \rfloor$ eigenvalues of B that have distance less then r from one of the elements of T.

Let us now build a bipartite graph, where the left set of nodes L contains the elements of $\Lambda(A)$, the right set of nodes R contains the elements of $\Lambda(B)$, and $\lfloor rn \rfloor$ additional nodes. Every additional node is connected to all the elements of L, and an element of $\Lambda(A)$ is connected to an element of $\Lambda(B)$ if and only if their distance is less then r. If we denote E the set of edges of the graph and N the set of its nodes, then we can define the neighborhood of a subset of nodes $P \subseteq N$ as

$$N(P) := \#\{u \in N : \exists v \in P, (v,u) \in E\}.$$

By using the previous derivations, we know that for any $T \subseteq L = \Lambda(A)$ it holds

$$N(T) \ge \#T - \lfloor rn \rfloor + \lfloor rn \rfloor = \#T.$$

Thanks to Hall's Marriage Theorem, that can be found for example in [7], there exists a matching for L, meaning that there exists an injective map $\alpha : L \to R$ such that

$$(u, \alpha(u)) \in E \qquad \forall u \in L.$$

Now let us consider the set

$$P := \{u \in L : \alpha(u) \in \Lambda(B)\}.$$

we know that $\#P \geq n - \lfloor rn \rfloor$, and we can enumerate the eigenvalues in $\Lambda(A) = L$ and $\Lambda(B)$ so that

$$\lambda_i(A) \in P, \quad \lambda_i(B) = \alpha(\lambda_i(A)) \qquad \forall\, i \leq n - \lfloor rn \rfloor.$$

Since u and $\alpha(u)$ are connected for all $u \in L$, we deduce that $\lambda_i(A)$ and $\lambda_i(B)$ are connected for at least $n - \lfloor rn \rfloor$ indices. By construction,

$$|\lambda_i(B) - \lambda_i(A)| < r \qquad \forall\, i \leq n - \lfloor rn \rfloor$$

so

$$\begin{aligned} d'(A,B) &= \min_{\sigma \in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |\Lambda(A) - \Lambda(B)_\sigma|_i^\downarrow \right\} \\ &\leq \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |\Lambda(A) - \Lambda(B)|_i^\downarrow \right\} \\ &< \frac{\lfloor rn \rfloor}{n} + r \leq 2r = 2\pi(A,B). \end{aligned}$$

This proves that for any $A, B \in fC^{n\times n}$ we have

$$\pi(A,B) \leq d'(A,B) \leq 2\pi(A,B).$$

Given now $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, we conclude

$$\pi(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \pi(A_n, B_n) \leq \limsup_{n\to\infty} d'(A_n, B_n) = d'(\{A_n\}_n, \{B_n\}_n),$$

$$d'(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} d'(A_n, B_n) \leq \limsup_{n\to\infty} 2\pi(A_n, B_n) = 2\pi(\{A_n\}_n, \{B_n\}_n).$$

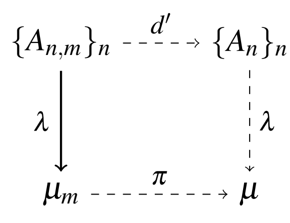
□

The two distances d' and π are equivalent, so they induce the same topology on the space $\mathscr{E}$ and they respect a property of closeness given by the following lemma.

Lemma 10. *Let $\{A_{n,m}\}_n \sim_\lambda \mu_m$, where $\{A_{n,m}\}_n \in \mathscr{E}$ and $\mu_m \in \mathbb{P}$ for every m. If we consider the statements below*

1. $\pi(\mu_m, \mu) \xrightarrow{m\to\infty} 0$,
2. $\{A_n\}_n \sim_\lambda \mu$,
3. $d'(\{A_{n,m}\}_n, \{A_n\}_n) \xrightarrow{m\to\infty} 0$,

where $\{A_n\}_n \in \mathscr{E}$ and $\mu \in \mathbb{P}$, then any two of them are true if and only if all of them are true.



Proof.
1.3. $\implies$ 2.) We know that

$$\pi(A_n,\mu) \le \pi(A_n,A_{n,m}) + \pi(A_{n,m},\mu_m) + \pi(\mu_m,\mu) \qquad \forall n,m.$$

Given $\varepsilon > 0$, we can find M such that

$$\pi(\mu_m,\mu) \xrightarrow{m\to\infty} 0 \implies \pi(\mu_m,\mu) < \varepsilon \quad \forall m > M,$$
$$d'(\{A_{n,m}\}_n,\{A_n\}_n) \xrightarrow{m\to\infty} 0 \implies d'(\{A_{n,m}\}_n,\{A_n\}_n) < \varepsilon \quad \forall m > M.$$

Using Lemma 9, we obtain

$$\limsup_{n\to\infty} \pi(A_{n,m},A_n) = \pi(\{A_{n,m}\}_n,\{A_n\}_n) \le d'(\{A_{n,m}\}_n,\{A_n\}_n).$$

We can then fix $m > M$ and find $N > 0$ such that

$$\pi(A_{n,m},A_n) \le 2\varepsilon, \quad \pi(A_{n,m},\mu_m) \le \varepsilon \qquad \forall n > N.$$

We obtain that

$$\pi(A_n,\mu) \le 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon \qquad \forall n > N,$$

and hence we conclude that

$$\pi(A_n,\mu) \xrightarrow{n\to\infty} 0 \implies \{A_n\}_n \sim_\lambda \mu.$$

2.3. $\implies$ 1.) Thanks to Lemma 8, we know that

$$\{A_{n,m}\}_n \sim_\lambda \mu_m, \quad \{A_n\}_n \sim_\lambda \mu \implies \pi(\mu_m,\mu) = \pi(\{A_{n,m}\}_n,\{A_n\}_n)$$

and, using with Lemma 9, we conclude that

$$\pi(\mu_m,\mu) = \pi(\{A_{n,m}\}_n,\{A_n\}_n) \le d'(\{A_{n,m}\}_n,\{A_n\}_n) \xrightarrow{m\to\infty} 0.$$

1.2. $\implies$ 3.) Thanks to Lemma 8, we know that

$$\{A_{n,m}\}_n \sim_\lambda \mu_m, \quad \{A_n\}_n \sim_\lambda \mu \implies \pi(\{A_{n,m}\}_n,\{A_n\}_n) = \pi(\mu_m,\mu)$$

and, using with Lemma 9, we conclude that

$$d'(\{A_{n,m}\}_n,\{A_n\}_n) \le 2\pi(\{A_{n,m}\}_n,\{A_n\}_n) = 2\pi(\mu_m,\mu) \xrightarrow{m\to\infty} 0.$$

□

4.2 Proofs of Theorems

We can finally prove that d' identifies two sequences if and only if they have the same spectral symbol.

Theorem 1. *If* $\{A_n\}_n \sim_\lambda f(x)$*, then*

$$\{B_n\}_n \sim_\lambda f(x) \iff d'(\{A_n\}_n, \{B_n\}_n) = 0.$$

Proof. Let μ be the probability measure associated to $f(x)$. Let also $\{A_{n,m}\}_n$ and μ_m be constant sequences defined as

$$A_{n,m} := A_n \qquad \forall n, m, \qquad\qquad \mu_m := \mu \qquad \forall m.$$

We know by hypothesis that

$$\{A_{n,m}\}_n \sim_\lambda \mu_m, \qquad \pi(\mu_m, \mu) \xrightarrow{m\to\infty} 0,$$

therefore, owing to Lemma 10, we obtain the equivalence

$$\{B_n\}_n \sim_\lambda \mu \iff d'(\{A_{n,m}\}_n, \{B_n\}_n) \xrightarrow{m\to\infty} 0,$$

which can be rewritten as

$$\{B_n\}_n \sim_\lambda f(x) \iff d'(\{A_n\}_n, \{B_n\}_n) = 0.$$

□

The other theorem shows that the GLT symbol represents in fact an ordering of the sequence eigenvalues. Given a sequence $\{A_n\}_n \in \mathscr{E}$ with a spectral symbol $k(x)$, we can consider the diagonal matrices $D_n \in \mathbb{C}^{n\times n}$ containing the eigenvalues of A_n. We get again that $\{D_n\}_n \sim_\lambda k(x)$, so we can focus only on diagonal sequences. A permutation of the eigenvalues is thus formalized as the similarity $P_n D_n P_n^T$ with P_n permutation matrices. In [3], we showed that a function $k(x) \otimes 1$ is a GLT symbol for a diagonal sequence $\{D_n\}_n$ if and only if the piecewise linear functions interpolating the ordered entries of D_n on $[0,1]$ converge in measure to $k(x)$. Thanks to the existence of the natural order on $\mathbb{R}$, we deduced that for any real diagonal sequence $\{D_n\}_n$, with a real spectral symbol $k(x)$, there exists a sequence of permutations $\{P_n\}_n$ such that

$$\{P_n D_n P_n^T\}_n \sim_{GLT} k(x) \otimes 1.$$

We could not extend the result on the complex plane, due to the lack of a natural ordering. Using the spectral measure theory we developed, we can now bypass the problem, since spectral symbols with the same distribution are now identified into an uniquely determined probability measure.

Theorem 2. *Given a measurable function $k : [0,1] \to \mathbb{C}$, and a diagonal sequence $\{D_n\}_n$ with spectral symbol $k(x)$, there exists a sequence $\{P_n\}_n$ of permutation matrices such that*

$$\{P_n D_n P_n^T\} \sim_{GLT} k(x) \otimes 1.$$

Proof. The space of continuous functions is dense in the space of measurable functions with the convergence in measure. Thus, there exist $k_m(x) \in C[0,1]$ that converge in measure to $k(x)$. Using Lemma 5, we can find a diagonal sequence $\{D'_n\}_n$ with

$$\{D'_n\}_n \sim_{GLT} k(x) \otimes 1, \qquad \{D'_n\}_n \sim_\lambda k(x).$$

Theorem 1 leads to $d'(\{D_n\}_n, \{D'_n\}_n) = 0$ and owing to Lemma 4, there exist permutation matrices $\{P_n\}_n$ such that

$$d_{acs}\left(\{D'_n\}_n, \{P_n D_n P_n^T\}_n\right) = 0.$$

Using the fact that the GLT space is closed for the pseudometric d_{acs}, and that the distance of the GLT symbols is equal to the distance of the sequences for Theorem 3, we conclude that $\{P_n D_n P_n^T\}_n \sim_{GLT} k(x) \otimes 1$. ☐

5 Future Works

The theory of spectral measures is still a work in progress, with open questions and many possible extensions.

For example, we have seen that the space of probability measures corresponds to the space of sequences which admit a spectral symbol, but the sequences admitting a general spectral measure (not necessarily a probability measure) is larger. The difference between 1 and the mass of a spectral measure can be interpreted as the rate of eigenvalues not converging to finite values, and consequentially we can admit spectral symbols $f : [0,1] \to \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the Riemann Sphere or the Alexandroff Compactification of $\mathbb{C}$. The insight on the sequences of matrices is that they may have a fraction of the asymptotic spectrum that diverge to ∞ in modulus, so a spectral symbol with values on $\mathbb{C}^*$ may also catch this new behaviour. The introduction of these new functions probably leads to a variation of Corollary 1, where a sequence admits a spectral measure if and only if it admits a spectral symbol with values on $\mathbb{C}^*$. The downside of this extension is that the distance π does not induce the vague convergence on the space of finite measures, so we need to find a new metric that mimics the characteristics of the Lèvy-Prokhorov metric.

All this document is focused on spectral symbols/measures, but the same analysis can be performed using the singular values instead of the eigenvalues, leading to a theory focused into singular value symbols/measures, that will probably have some deep bounds with the GLT symbols. They are similar since both the GLT symbol and the singular value symbol of a sequence are unique, but at the same time they are also very different since the space of measures lacks a group structure, and two sequences with different GLT symbols may have the same singular value symbol.

Eventually, it seems that spectral measures arise naturally even in algebraic geometry (see, for example, [13]) so further connections can be also developed in different areas of mathematics.

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